Eessential surfaces in the exteriors of torus knots with twists on 2-strands

by

Kanji Morimoto

Department of IS and Mathematics, Konan University
Higashi-Nada, Okamoto 8-9-1, Kobe 658-8501, Japan
e-mail : morimoto@konan-u.ac.jp

Abstract. In the present paper, we will show that torus knots with twists on 2-strands contain no closed essential surfaces in the exteriors. As an application, we will show that Hoiden-Morimoto's inequality for 1-bridge genus of knots is best possible.

Key words and phrases : torus knots, small knots, meridionally small knots.
2000 Mathematics Subject Classification : 57M25, 57N10

1. Introduction

Let K be a knot in the 3-sphere S^3, $N(K)$ the regular neighborhood of K in S^3 and $E(K) = cl(S^3 - N(K))$ the exterior. Let F be a surface (i.e. a connected 2-manifold) properly embedded in $E(K)$. Then we say that F is a closed essential surface if F is closed, incompressible and not parallel to the torus $\partial E(K)$, and that F is a meridionally essential surface if $\partial F \neq \emptyset$, each component of ∂F is a meridian of $N(K)$, F is incompressible and not parallel to an annulus in $\partial E(K)$. We say that a knot K is small if $E(K)$ contains no closed essential surfaces, and that a knot K is meridionally small if $E(K)$ contains no meridionally essential surfaces. We note that if a knot K in S^3 is small then it is meridionally small by [CGLS, Theorem 2.0.3].

For coprime integers p, q, we denote the torus knot of type (p, q) by $T(p, q)$. Let r be an arbitrary integer, then we consider the knot obtained from $T(p, q)$ by adding r-times full twists at mutually parallel 2-strands to $T(p, q)$ as illustrated in Figure 1, and denote it by $K(p, q; r)$ (c.f. [MSY]). For the definition of torus knots we refer [Ro]. In the present paper, we will show :

Theorem 1.1 *For any coprime integers p, q, and for any integer r, $K(p, q; r)$ is small and meridionally small.*

Remark 1 Since small knots are meridionally small as we mentioned above, it is sufficient to show that $K(p, q; r)$ is small. However, by some technical reason, we will show the smallness and the meridional smallness simultaneously.

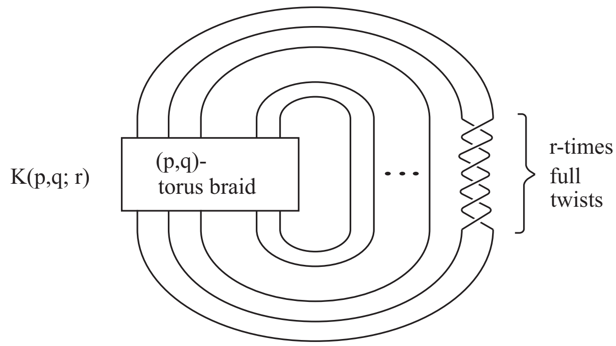


Figure 1

Remark 2 In the present paper, we deal with torus knots with twists on 2-strands. In general, we can deal with torus knots with twists on arbitrary strands. In the paper [MY], we deal with such knots, and show that, for any composite number n, there are infinitely many torus knots with twists on n-strands containing essential tori in the exteriors([MY, Theorem 2.2]).

We say that (V_1, V_2) is a Heegaard splitting of S^3 if $S^3 = V_1 \cup V_2$, $V_1 \cap V_2 = \partial V_1 = \partial V_2$ and both V_1 and V_2 are handlebodies. The genus of V_1 (= the genus of V_2) is called the genus of the Heegaard splitting and the surface $\partial V_1 = \partial V_2$ is called the Heegaard surface of the Heegaard splitting. Then for any knot K in S^3 it is well known that there is a Heegaard splitting (V_1, V_2) of S^3 such that K intersects V_i in a single trivial arc in V_i for both $i = 1, 2$. Hence we define the 1-bridge genus $g_1(K)$ of K as the minimal genus among all such Heegaard splittings (V_1, V_2) of S^3. For two knots K_1, K_2 in S^3, we denote the connected sum of K_1 and K_2 by $K_1 \# K_2$. Then by a little observation, we immediately see the following :

Fact 1.2 *For any two knots K_1 and K_2 in S^3, $g_1(K_1 \# K_2) \leq g_1(K_1) + g_1(K_2)$.*

On the problem to estimate the lower bound of $g_1(K_1 \# K_2)$, Hoiden and the author showed :

Theorem 1.3 ([Ho, Theorem], [M3, Theorem 3.10]) *Let K_1, K_2 be two knots in S^3. If K_1 and K_2 are meridionally small, then $g_1(K_1 \# K_2) \geq g_1(K_1) + g_1(K_2) - 1$.*

As an application of Theorem 1.1, we will show that the above inequality is best possible as follows (note that smallness implies meridionally smallness) :

Theorem 1.4 *There are infinitely many pairs of small knots K_1, K_2 in S^3 with $g_1(K_1 \# K_2) = g_1(K_1) + g_1(K_2) - 1$.*

Proof Let $t(K)$ be the tunnel number of a knot K, i.e., $t(K)$ is the minimum number of arcs properly embedde in $E(K)$ such that the exterior of the arcs is homeomorphic to a handlebody. Then it is well known that the inequality $t(K) \leq g_1(K) \leq t(K)+1$ holds for any knot K. To show Theorem 1.4, we need the following lemma.

Lemma 1.5 ([M2, Proposition 1.7]) *Let K be a knot with $g_1(K) = t(K) + 1$. Then $g_1(K \# K') \leq g_1(K)$ holds for any 2-bridge knot K'.*

Now, let m be an integer and consider the knot $K_1 = K(7, 17, 5m-2)$, i.e., consider the case when $p = 7, q = 17$ and $r = 5m - 2$. Then, by Theorem 1.1, K_1 is small. Moreover, by [MSY, Theorem 2.1], $t(K_1) = 1$ and $g_1(K_1) = 2$, i.e., $g_1(K_1) = t(K_1)+1$. Let K_2 be a (non-trivial) 2-bridge knot in S^3. Then it is well known that K_2 is small and $g_1(K_2) = 1$. Then by Lemma 1.5, $g_1(K_1 \# K_2) \leq g_1(K_1) = 2$. On the other hand, $g_1(K_1 \# K_2) \geq 2$ because 1-bridg genus one knots are tunnel number one knots, and hence prime by [No, Sc]. Thus $g_1(K_1 \# K_2) = 2$ and $g_1(K_1 \# K_2) = g_1(K_1)+g_1(K_2)-1$ for the small knots K_1 and K_2. This complets the proof of Theorem 1.4. □

By the way, for the additivity of tunnel number, it is well known that $t(K_1 \# K_2) \leq t(K_1) + t(K_2) + 1$. Concerning the best possibility of this inequality, by putting $K_m = K(7, 17, 5m-2)$ and $K_n = K(7, 17, 5n-2)$, we have shown in [MSY] that $t(K_m) = t(K_n) = 1$ and $t(K_m \# K_n) = 3$, i.e., $t(K_m \# K_n) = t(K_m) + t(K_n) + 1$. Moreover, related to this problem, we have shown the following theorem in [M1].

Theorem 1.6 ([M1, Theorem 1.6]) *Let K_1 and K_2 be both meridionally small knots. Then $t(K_1 \# K_2) = t(K_1)+t(K_2)+1$ if and only if $g_1(K_i) = t(K_i)+1$ for both $i = 1, 2$.*

By Theorem 1.1, we see that both K_m and K_n are meridionally small, and this shows that there are infinitely many pairs of knots satisfying the hypothesis and the conclusion of the above theorem.

Throughout the present paper, we will work in the piecewise linear category. For a manifold X and a subcomplex Y of X, we denote a regular neighborhood of Y in X by $N(Y, X)$ or simply $N(Y)$.

2. Preliminaries for the proof of Theorem 1.1

Let V_1 be the standard solid torus in S^3, and take a torus knot $T(p, q)$ in ∂V_1 so that it winds around V_1 with p-times in the longitudal direction and q-times in the meridional direction. Let D_1 be a 2-disk, and put $V_2 = D_1 \times I$ with $D_1 = D_1 \times \{\frac{1}{2}\}$, $D_1^0 = D_1 \times \{0\}$ and $D_1^1 = D_1 \times \{1\}$, where $I = [0, 1]$ is the unit interval. Remove parallel two subarcs from $T(p, q)$ and attach the 1-handle V_2 to V_1 so that the two subarcs removed can be regarded as two arcs in ∂V_2. Then we can regard $T(p,q)$ as a knot in the genus two surface $\partial(V_1 \cup V_2)$ such that the knot intersects D_1 in exactly two points. Perform r-times full twists at D_1, then we get the knot $K(p,q;r)$ in $\partial(V_1 \cup V_2)$ as in Figure 2.

Put $K = K(p,q;r)$ and we may assume $p > 0$. If $r = 0$, then K is the torus knot $T(p,q)$ and is small (and meridionally small). Hence hereafter, we assume $r \neq 0$. Moreoveer, if $r < 0$, then by taking $K(p,-q;-r)$ instead of $K(p,q;r)$ we may assume that $r > 0$ because $E(K(p,-q;-r))$ is homeomorphic to $E(K(p,q;r))$. Let D_2 be a 2-disk properly embedded in $cl(S^3 - (V_1 \cup V_2))$ such that $D_1 \cap D_2 =$ a point and $D_2 \cap (K - V_2) = \emptyset$ as in Figure 2. Put $\partial D_2 = \gamma_1 \cup \gamma_2$ such that $\gamma_1 = \partial D_2 \cap V_1$ and $\gamma_2 = \partial D_2 \cap V_2$. By these constructions and definitions, we see that $K \cap D_1 = K \cap \partial D_1 =$ two points, and $K \cap D_2 = K \cap \partial D_2 = K \cap \gamma_2 = 2r$ points as in Figures 2 and 3.

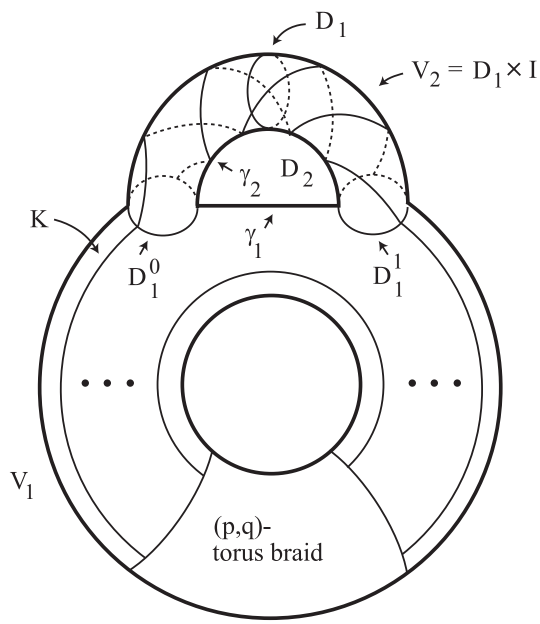


Figure 2

Suppose K is not small or not meridionally small, then there is a closed essential surface or a meridionally essential surface properly embedded in $E(K)$. If the surface is closed, then denote it by F. If the surface is not closed, then by capping the boudary components with meridian disks of $N(K)$, we get a closed surface and denote it by F. Hence we see that if $F \cap K = \emptyset$ then $F - K = F$ is incompressible in $S^3 - K$ and not a torus parallel to $\partial E(K)$, and that if $F \cap K \neq \emptyset$ then $F - K$ is incompressible in $S^3 - K$ and F is not a 2-sphere bounding a 3-ball intersecting K in a trivial arc.

Lemma 2.1 *Suppose there is a disk D in S^3 such that $D \cap K$ is a single point* *and $D \cap F = \partial D$ is essential in F or bounds such a disk D' in F that the 2-sphere $D \cup D'$ does not bound a 3-ball intersecting K in a trivial arc. Let F' be a closed surface(s) obtained from F by a compression along D. Then $F' \cap K \neq \emptyset$, $F' - K$ is incompressible in $S^3 - K$, and each component of F' is not a 2-sphere bounding a 3-ball intersecting K in a trivial arc.*

Proof. If $F' - K$ is compressible in $S^3 - K$, then there is a compressing disk of $F' - K$, say D', such that $D' \cap K = \emptyset$ and $D' \cap F' = \partial D'$ is an essential loop in $F' - K$. Then, since F is obtained from F' by tubing along a subarc of K, D' is a compressing disk of $F - K$ in $S^3 - K$, a contradiction. If F' is a 2-sphere bounding a 3-ball intersecting K in a trivial arc, then F is a torus parallel to $\partial E(K)$, a contradiction. If F' consists of two components and at lease one component is a 2-sphere bounding a 3-ball intersecting K in a trivial arc, then this means that ∂D bounds such a disk D' in F that the 2-sphere $D \cup D'$ bounds a 3-ball intersecting K in a trivial arc, a contradiction. This completes the proof of Lemma 2.1. □

Now, perform the compression as many as possible if F has disks as in Lemma 2.1, and we denote (a component of) the resulting closed surface(s) by F again. Then, by Lemma 2.1, F is a closed surface in S^3 such that $cl(F - N(K))$ is essential in $E(K)$, and hence we say that $F - K$ is essential in $S^3 - K$.

By the incompressibility of $F - K$ in $S^3 - K$, we may assume that $D_1 \cap F$ consists of n arcs for some $n \geq 0$ and $V_2 \cap F$ consists of n rectangles, where each arc of $D_1 \cap F$ separates the two points $D_1 \cap K = \partial D_1 \cap K$. We assume that n is minimal among all such surfaces F. Put $V_3 = cl(S^3 - (V_1 \cup V_2))$. Then V_3 is a genus two handlebody and $(V_1 \cup V_2, V_3)$ is a genus two Heegaard splitting of S^3. Put $V_1 \cap F = F_1$, $V_2 \cap F = F_2$ and $V_3 \cap F = F_3$. Put $\partial\gamma_1 = \partial\gamma_2 = \{p_1, p_2\}$, and let $d_i (i = 1, 2)$ be the subarc of γ_2 such that $\partial d_i = \{p_i, q_i\}$ with $d_i \cap K = q_i$ as illustrated in Figure 3.

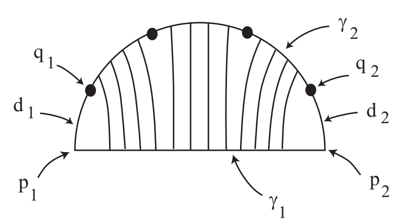


Figure 3

Lemma 2.2 *We may assume that each component of $F_3 \cap D_2$ is an arc connecting γ_2 and γ_1, and is disjoint from $d_1 \cup d_2$ as in Figure 3.*

Proof. By the incompressibility of $F - K$ in $S^3 - K$, we may assume that each component of $F_3 \cap D_2$ is an arc. If there is a component of $F_3 \cap D_2$ such that at least one end point is in d_1 or in d_2, then we can slide the end point along the arc d_1 or d_2 through p_1 or p_2 until the interior of γ_1. This sliding can be realized by an ambient isotopy of $F - K$ in $S^3 - K$. Hence we may assume that $d_1 \cup d_2$ has no point of $F_3 \cap \partial D_2$. At this stage, by performing a suitable deformation of $F - K$ after the above sliding if necessary, we may assume that $F_2 = V_2 \cap F$ consists of rectangles.

Let α be an outermost arc component of $F_3 \cap D_2$ in D_2, Δ the outermost disk and put $\beta = cl(\partial\Delta - \alpha)$. If $\partial\alpha \subset \gamma_1$, then $\beta \subset \gamma_1$ and we can eliminate the arc α by sliding along Δ. Suppose $\partial\alpha \subset \gamma_2$, then $\beta \subset \gamma_2$. First suppose $\beta \cap K = \emptyset$. Then β connects two adjacent rectangles of F_2, say R_1 and R_2. Perform a boundary compression of F_3 along Δ, and let b be the band in V_2 produced by the boundary compression . Then, since $R_1 \cup b \cup R_2$ is a ∂-parallel disk in V_2 and the parallelism is disjoint from K, we can push $R_1 \cup b \cup R_2$ out from V_2. This decrease the number n, a contradiction.

Next suppose $\beta \cap K \neq \emptyset$. Then $\beta \cap K$ consists of a single point of $K \cap \gamma_2$, because each component of F_2 separates the two components of $K \cap V_2$. Then β meets a single rectangle of F_2, say R, and let Δ' be the boundary compressing disk of R in V_2 with $\Delta' \cap \partial V_2 = \beta$. Then $D = \Delta \cup \Delta'$ is a disk such that $D \cap F = \partial D$ and D intersects K in a single point. Since F has no disk as in Lemma 2.1, ∂D bounds a disk in F, say D', such that $D \cup D'$ bounds a 3-ball intersecting K in a trivial arc. Hence by sliding F along the 3-ball, we can eliminate the rectangle R. This contradicts the minimality of n. After all, we see that each component α of $F_3 \cap D_2$ is an arc connecting γ_2 and γ_1 with $\alpha \cap (d_1 \cup d_2) = \emptyset$. This completes the proof of Lemma 2.2. □

Let $K \cap V_1 = K \cap \partial V_1 = k_1 \cup k_2$ be two subarcs of K and put $\partial k_1 = \{x_1, y_1\}$, $\partial k_2 = \{x_2, y_2\}$ so that $D_1^0 \cap (k_1 \cup k_2) = \{x_1, x_2\}$ and $D_1^1 \cap (k_1 \cup k_2) = \{y_1, y_2\}$. A schematic picture of $\{k_1, k_2, D_1^0, D_1^1, \gamma_1\}$ in ∂V_1 is illustrated in Figure 4.

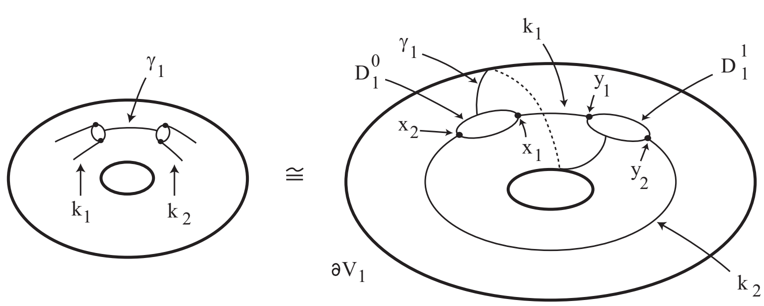


Figure 4

Lemma 2.3 *We may assume the following* :

(1) *There is no pair of a subarc α of $k_1 \cup k_2$ and an arc β properly embedded in F_1*

such that $\alpha \cap \beta = \partial\alpha = \partial\beta$ *and* $\alpha \cup \beta$ *bounds a disk in* V_1.
(2) *There is no* 2*-gon in* $\partial F_1 \cup (k_1 \cup k_2)$ *which bounds a disk in* ∂V_1.
(3) *There is no* 2*-gon in* $\partial F_1 \cup \gamma_1$ *which bounds a disk in* ∂V_1.
(4) *There is no* 2*-gon in* $\partial F_1 \cup ((\partial D_1^0 \cup \partial D_1^1) - \{x_1, x_2, y_1, y_2\})$ *which bounds a disk in* ∂V_1.

Proof. (1) Suppose there is such a pair (α, β), and let Δ be the disk in V_1 with $\partial\Delta = \alpha \cup \beta$. Let $N(\Delta)$ be a regular neighborhood of Δ in S^3 such that $N(\Delta) \cap F$ is a disk which is a regular neighborhood of β in F, say $N(\beta, F)$. Put $c = \partial N(\beta, F)$, then since c is a loop in $\partial N(\Delta)$ and $N(\beta, F) \cap K =$ two points, c bounds a disk in $\partial N(\Delta)$ disjoint from K. By the incompressibility of $F - K$ in $S^3 - K$, c bounds a disk, say B, in $F - K$. Then $F = B \cup N(\beta, F)$ is a 2-sphere bounding a 3-ball intersecting K in a trivial arc, a contradiction.

(2) If there is such a 2-gon in $\partial F_1 \cup (k_1 \cup k_2)$, then we can find a pair (α, β) satisfying the condition (1), a contradiction.

(3) Suppose there is a 2-gon $\alpha \cup \beta$ in $\partial F_1 \cup \gamma_1$ with $\alpha \subset \partial F_1$ and $\beta \subset \gamma_1$. Then by the ambient isotopy along the disk bounded by $\alpha \cup \beta$, F is deformed so that $F_3 \cap D_2$ contains an arc whose end points are in γ_2. Then by the argument in the proof of Lemma 2.2, we have a contradiction.

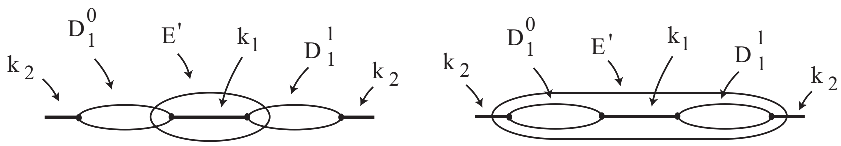
(4) Suppose there is a 2-gon $\alpha \cup \beta$ such that $\alpha \subset \partial F_1$ and $\beta \subset ((\partial D_1^0 \cup \partial D_1^1) - \{x_1, x_2, y_1, y_2\})$. If $\alpha \subset D_1^0$ (or D_1^1), then we have a contradiction because each component of $\partial F_1 \cap D_1^0$ (or $\partial F_1 \cap D_1^1$ resp.) is an arc which separates x_1 and x_2 (or y_1 and y_2 resp.). Suppose $\alpha \cap (D_1^0 \cup D_1^1) = \partial\alpha$, and let Δ be the disk in ∂V_1 with $\partial\Delta = \alpha \cup \beta$. Then by the ambient isotopy of F along Δ, we get a band in V_2 connecting two rectangles in F_2. Then by the argument in the proof of Lemma 2.2, we have a contradiction. This completes the proof of Lemma 2.3. □

By the incompressibility of F_1 in V_1, each component of F_1 is a ∂-parallel disk, a ∂-parallel annulus or a meridian disk of V_1.

Lemma 2.4 *Let* E *be a* ∂*-parallel disk component of* F_1*, and* E' *the disk in* ∂V_1 *to which* E *is parallel. Then we may assume that one of the following folds* :
(1) E' *is a small regular neighborhood of* k_i *in* ∂V_1 $(i = 1, 2)$ (*Figure* 5(1)),
(2) E' *is a small regular neighborhood of* $D_1^0 \cup k_i \cup D_1^1$ *in* ∂V_1 $(i = 1, 2)$ (*Figure* 5(2)).

Proof. Put $X = \{x_1, x_2, y_1, y_2\}$, and we divide the proof into the following five cases.

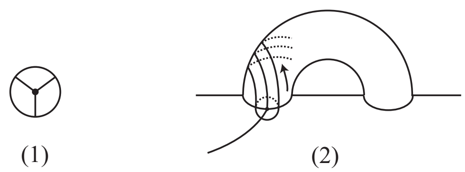
Case (0) : $E' \cap X = \emptyset$. In this case, if $E' \cap (D_1^0 \cup D_1^1 \cup k_1 \cup k_2 \cup \gamma_1) \neq \emptyset$, then there is a 2-gon in $\partial E' \cap ((\partial D_1^0 \cup \partial D_1^1 \cup k_1 \cup k_2 \cup \gamma_1) - X)$. This contradicts Lemma 2.3. Thus $E' \cap (D_1^0 \cup D_1^1 \cup k_1 \cup k_2 \cup \gamma_1) = \emptyset$ and we can eliminate the disk E.



(1) (2)

Figure 5

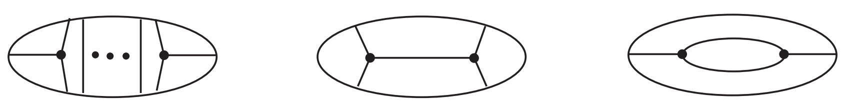
Case (1) : $E' \cap X =$ one point. In this case, by Lemma 2.3, $E' \cap (\partial D_1^0 \cup \partial D_1^1 \cup k_1 \cup k_2)$ is a wheel with three edges as in Figure 6(1). Then by the deformation along the arrow indicated in Figure 6(2), we can decrease the number n, a contradiction.



(1) (2)

Figure 6

Case (2) : $E' \cap X =$ two points. In this case, by Lemma 2.3, $E' \cap (\partial D_1^0 \cup \partial D_1^1 \cup k_1 \cup k_2)$ is one of the three patterns illustrated in Figure 7, i.e., (i) the two points are separated, (ii) the two points are connected by an arc, (iii) the two points are connected by two arcs. In addition, by Case (1), we may assume that E' contains no other disks.



(i) (ii) (iii)

Figure 7

In case (i), by a boundary compression of E, we get a ∂-parallel disk satisfying the condition in Case (1). Then we get a contradiction as in Case (1). In case (ii), if the arc connecting the two points is a subarc of $\partial(D_1^0 \cup D_1^1)$, then this is a contradiction because each arc of $D_1^0 \cap F_1$ (or $D_1^1 \cap F_1$) separates the two points x_1 and x_2 (or y_1 and y_2 resp.). Hence the arc connecting the two points is k_1 or k_2, and we get the conclusion (1). In case (iii), the circle made by the two arcs is ∂D_1^0 or ∂D_1^1. Then we have $n = 0$ because $\partial D_1^0 \cap F_1 = \emptyset$ or $\partial D_1^1 \cap F_1 = \emptyset$. On the other hand, we have $n > 0$ because γ_1 connects ∂D_1^0 and ∂D_1^1 as in Figure 4 and γ_1 intersects $\partial E' = \partial E$. This is a contradiction.

Case (3) $E' \cap X =$ three points. If $E' \cap (\partial D_1^0 \cup \partial D_1^1 \cup k_1 \cup k_2)$ is not connected, then by the argument in case (i) of Case (2), we have a contradiction. Hence $E' \cap (\partial D_1^0 \cup \partial D_1^1 \cup k_1 \cup k_2)$ is connected and one of the two patterns illustrated in Figure 8 happens. In addition,we may assume that $\{x_1, x_2\}$ is contained in E'.

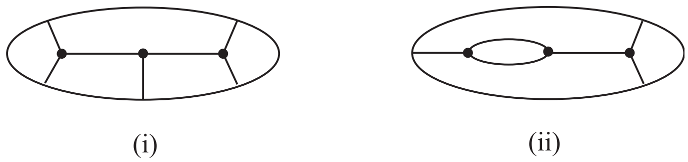


Figure 8

In case (i), one of the two arcs of $\partial D_1^0 - \{x_1, x_2\}$ is contained in E'. Then the other arc and ∂F_1 makes a 2-gon in $\partial F_1 \cup \partial D_1^0$. This contradicts Lemma 2.3. In case (ii), ∂D_1^0 is contained in E'. Since each component of F_1 meeting D_1^0 is a disk satisfying the conclusion (1) of this lemma, there are n such disk components because of $|D_1^0 \cap F_1| = n$. On the other hand, E meets D_1^1 and this shows that $|D_1^1 \cap F_1| > n$. This contradicts that $|D_1^0 \cap F_1| = |D_1^1 \cap F_1| = n$.

Case (4) $E' \cap X$ = four points. If $E' \cap (\partial D_1^0 \cup \partial D_1^1 \cup k_1 \cup k_2)$ is not connected, then by the arguments in Cases (1), (2), (3), we can regard E' as $N(k_1) \cup N(k_2) \cup b$, where $N(k_i)$ is a small regular neighborhood of k_i in $\partial V_1 (i = 1, 2)$ and b is a band with $b \cap (\partial D_1^0 \cup \partial D_1^1 \cup k_1 \cup k_2) = \emptyset$. Then by performing a boundary compression of F along b, E is changed to two disk components of F_1 both of which satisfy the conclusion (1) of this lemma. Suppose $E' \cap (\partial D_1^0 \cup \partial D_1^1 \cup k_1 \cup k_2)$ is connected. Then we have one of the five patterns illustrated in Figure 9.

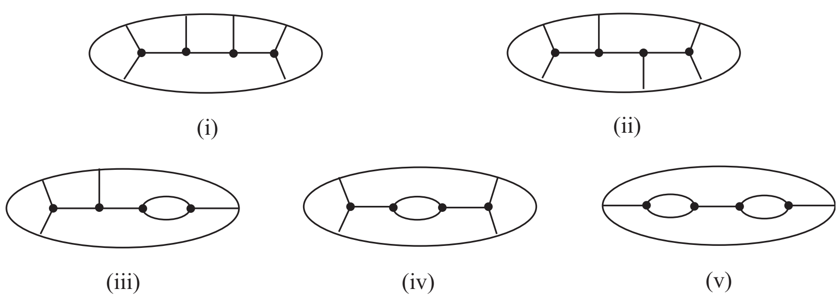


Figure 9

In cases (i), (ii), we have a contradiction as in case (i) of Case (3). In cases (iii), (iv), we have a contradiction as in case (ii) of Case (3). Hence we are in case (v) and E is a disk satisfying the conclusion (2) of this lemma. This completes the proof of Lemma 2.4. □

Put $cl(K - (k_1 \cup k_2)) = k_1' \cup k_2'$, i.e., k_1' and k_2' are the two arcs $K \cap V_2 = K \cap \partial V_2$

such that k_1' connects x_1 and y_2 and k_2' connects x_2 and y_1. Hereafter, by changing the letters if necessary, we assume that q_1 is contained in k_2' and q_2 is contained in k_1'. Moreover, we put $\partial D_1^0 = a_1 \cup b_1 \cup c_1$ with $\partial a_1 = \{x_1, p_1\}$, $\partial b_1 = \{p_1, x_2\}$, $\partial c_1 = \{x_1, x_2\}$ and $\partial D_1^1 = a_2 \cup b_2 \cup c_2$ with $\partial a_2 = \{y_1, p_2\}$, $\partial b_2 = \{p_2, y_2\}$, and $\partial c_2 = \{y_1, y_2\}$ as illustrated in Figure 10.

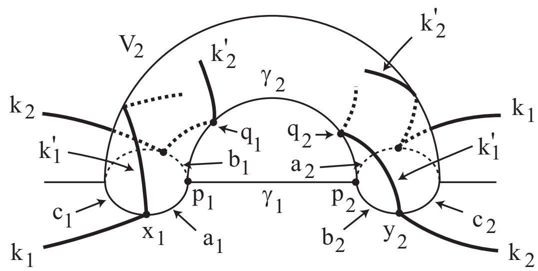


Figure 10

Lemma 2.5 (1) *Any component of $F_1 \cap D_1^0$ (or $F_1 \cap D_1^1$) is an arc connecting a point in a_1 (or a_2 resp.) and a point in c_1 (or c_2 resp.).*
(2) *There is no 3-gon which consists of a subarc of a_1 (or a_2), a subarc of ∂F_1 and a subarc of γ_1 such that it bounds a disk in ∂V_1.*
(3) *There is no 3-gon which consists of a subarc of ∂D_1^0 (or ∂D_1^1), a subarc of ∂F_1 and a subarc of $k_1 \cup k_2$ such that it bounds a disk in ∂V_1.*

Proof. (1) Since each component of $F_2 = F \cap V_2$ is a rectangle separating k_1' and k_2', any component of $F_1 \cap D_1^0$ is an arc connecting a point in $a_1 \cup b_1$ and a point in c_1. Suppose there is an arc component of $F_1 \cap D_1^0$ connecting a point in b_1 and a point in c_1. This means that there is a subarc of ∂F_2 which connects a point in b_1 and a point in d_1. But d_1 does not meet F by Lemma 2.2. This contradiction shows that any component of $F_1 \cap D_1^0$ is an arc connecting a point in a_1 and a point in c_1. Similarly, we see that any componento of $F_1 \cap D_1^1$ is an arc connecting a point in a_2 and a point in c_2.

(2) Suppose there is such a 3-gon, and let Δ be the disk in ∂V_1 bounded by the 3-gon. Then by an ambient isotopy along Δ, we can deform F so that $F \cap D_2$ has an arc component whose end points are in γ_2 as illustrated in Figure 11. Then by the argument in the proof of Lemma 2.2, we decrease the number n, a contradiction.

(3) Suppose there is such a 3-gon, and let Δ be the disk in ∂V_1 bounded by the 3-gon. Then by an ambient isotopy along Δ, i.e., a boundary compression for F_1 and a sliding along the arrow indicated in Figure 12, we can decrease the number n, a contradiction. This completes the proof of Lemma 2.5. □

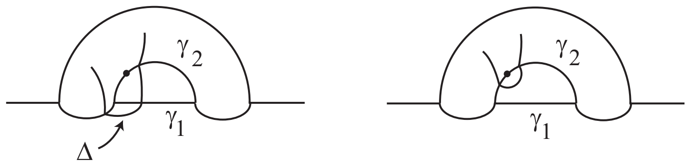


Figure 11

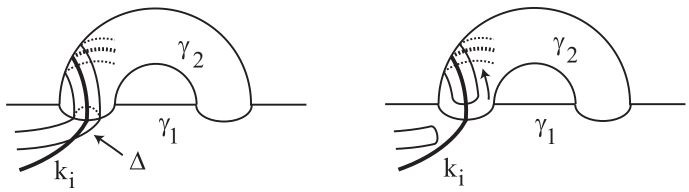


Figure 12

Lemma 2.6 *Let E be a ∂-parallel disk component of F_1, and E' the disk in ∂V_1 to which E is parallel. Then E' is a small regular neighborhood of k_1 in ∂V_1. This means that in the four conclusions in Lemma* 2.4 $((1),(2),(i=1,2))$, *the conclusion* $(1)(i=1)$ *only occurs.*

Proof. Suppose we are in the conclusion (1) of Lemma 2.4. If E' is a regular neighborhood of k_2, then there is an arc component of $D_1^0 \cap F_2$ connecting a point in b_1 and a point in c_1 or connecting a point in a_1 and a point in c_1 with a 3-gon as in Lemma 2.5(2). This contradicts Lemma 2.5, and hence E' is a regular neighborhood of k_1.

Next suppose we are in the conclusion (2) of Lemma 2.4, and let E' be a regular neighborhood of $D_1^1 \cup k_i \cup D_1^1$ $(i=1,2)$. Since $\partial E \cap \gamma_1 \neq \emptyset$, we see that $F \cap D_2 \neq \emptyset$ and $n > 0$. If E' is a regular neighborhood of $D_1^0 \cup k_2 \cup D_1^1$, then there is a ∂-parallel disk component of F_1 which is parallel to a regular neighborhood of k_2 because $n > 0$. This contradicts that there is no such a disk as shown above. Thus E' is a regular neighborhood of $D_1^0 \cup k_1 \cup D_1^1$. Then, by the same reaseon as above, there is a ∂-parallel disk component of F_1, say E_1, which is parallel to a regular neighborhood of k_1. Since $\partial E \cap \gamma_1 \neq \emptyset$, there is an arc component of $F \cap D_2$, say α, which connects a point in ∂E and a point in γ_2. Then by sliding α (in fact by some ambient isotopy) as illustrated in Figure 13, we get a band b connecting E and E_1. Then the disk $E \cup b \cup E_1$ is parallel to a disk in ∂V_1 which satisfies the condition of Case (2)(i) in the proof of Lemma 2.4. This is a contradiction and completes the proof of Lemma 2.6. □

Lemma 2.7 *Let A be a ∂-parallel annulus component of F_1, and A' the annulus in*

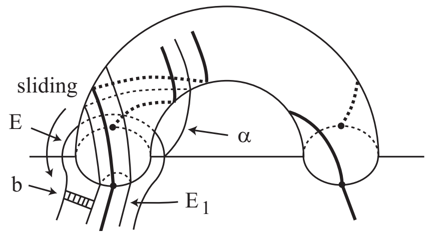


Figure 13

∂V_1 to which A is parallel. Then we may assume that A is a small regular neighborhood of $D_1^0 \cup k_1 \cup D_1^1 \cup k_2$.

Proof. Suppose there is an essential arc properly embedded in A', say e, with $e \cap (\partial D_1^0 \cup k_1 \cup \partial D_1^1 \cup k_2) = \emptyset$. Then by a boundary compression of A through e, we get a ∂-parallel disk. Then the disk satisfies the conclusion of Lemma 2.6, and by taking the number of annuli to be minimal, we may assume that there is no such an essential arc. Thus $A' \cap (\partial D_1^0 \cup k_1 \cup \partial D_1^1 \cup k_2)$ contains a central loop of A'. If one of ∂D_1^0 and ∂D_1^1 is a central loop of A', say ∂D_1^0, then $F_1 \cap D_1^0$ contains a loop component of $\partial A'$. This contradicts Lemma 2.5(1). Hence it is not ∂D_1^0 or ∂D_1^1, and hence it is $k_1 \cup k_2 \cup$ (a subarc of ∂D_1^0)$\cup$ (a subarc of ∂D_1^1). If $\partial D_1^0 \not\subset A'$ or $\partial D_1^1 \not\subset A'$, then we can find a 2-gon in $\partial A' \cup ((\partial D_1^0 \cup \partial D_1^1) - \{x_1, x_2, y_1, y_2\})$. This contradicts Lemma 2.3, and shows that $(\partial D_1^0 \cup k_1 \cup \partial D_1^1 \cup k_2) \subset A'$. This completes the proof of Lemma 2.7. □

3. Proof of Theorem 1.1

Since F_1 is incompressible in V_1, each component of F_1 is a ∂-parallel disk which is a small regular neighborhood of k_1 as in Lemma 2.6, a ∂-parallel annulus which is a small regular neighborhood of $D_1^0 \cup k_1 \cup D_1^1 \cup k_2$ as in Lemma 2.7 or a meridian disk of V_1. Hence we have the following two subcases.

Case I : $F_1 = \tilde{E} \cup \tilde{A}$, where $\tilde{E}$ consists of ∂-parallel disks and $\tilde{A}$ consists of ∂-parallel annuli.

Case II : $F_1 = \tilde{E} \cup \tilde{G}$, where $\tilde{E}$ consists of ∂-parallel disks and $\tilde{G}$ consists of meridian disks.

Suppose we are in Case I. If $\tilde{A} = \emptyset$, then $F_1 \cap \gamma_1 = \emptyset$ and $F_2 = \emptyset$. Hence $\tilde{E} = \emptyset$ and $F_1 = \emptyset$. This means that $F \cap K = \emptyset, F = F_3$ and $F \subset V_3$. Then F is compressible, and this is a contradiction. Thus $\tilde{A} \neq \emptyset$, and then $\tilde{E} \neq \emptyset$ because $F_2 \neq \emptyset$ and $n > 0$. Therefore we can put $\tilde{E} = E_1 \cup E_2 \cup \cdots \cup E_n$ and $\tilde{A} = A_1 \cup A_2 \cup \cdots \cup A_m$ for some m.

Since $D_2 \cap (V_1 \cup V_2) = \partial D_2$, we can take a product space $D_2 \times I$ in V_3 so that

$(D_2 \times I) \cap (V_1 \cup V_2) = \partial D_2 \times I$. Put $W_1 = (V_1 \cup V_2) \cup (D_2 \times I)$ and $W_2 = cl(S^3 - W_1) = cl(V_3 - (D_2 \times I))$. Then W_i $(i = 1, 2)$ is a solid torus and (W_1, W_2) is a genus one Heegaard splitting of S^3. Put $H_1 = F \cap W_1$ and $H_2 = F \cap W_2$. Since each component of ∂H_1 is a torus knot on ∂W_1 in S^3, in the following, we detect the torus knot type.

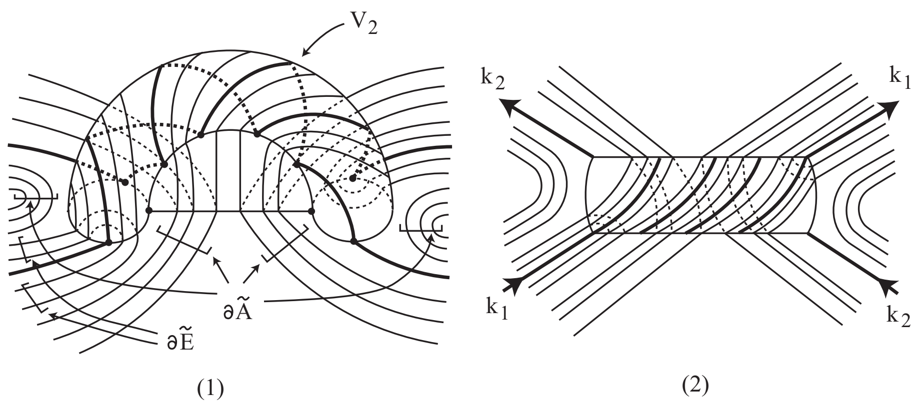


Figure 14

First, by a little observation, we have $2m = n(2r-1)$ with $n = 2k$ for some integer $k > 0$. Figure 14(1) illustrates the local intersection of $F_1 \cup F_2$ and $D_2 \cup \partial(V_1 \cup V_2)$ in the case when $n = 2, (k = 1), m = 3$ and $r = 2$, and Figure 14(2) shows the bird's eye picture of $\partial H_1 \cap \partial W_1$ from the higher position above V_2. Then, by this picture, we see that ∂H_1 consists of two loops in this case. If $n = 2k > 2$ (i.e., $k > 1$), then by considering the single loop in Figure 14(2) as a bunch of k loops, we can see that ∂H_1 consists of $n = 2k$ loops which correspond to n points $\partial \tilde{E} \cap c_1$. Since $k_1 \cup c_1 \cup k_2 \cup c_2$ is a simple closed curve in the torus ∂W_1, we can put $k_1 \cup c_1 \cup k_2 \cup c_2 = T(a, b)$ for some coprime integers (a, b). Then, since $T(a, b)$ is obtained from the given torus knot $T(p, q)$ by a self-fusion along an arc which connects two points in parallel two strands of $T(p, q)$ and intersects γ_1 in a single point, we have $pb - qa = \pm 2$ by [M4, Theorem 1.2], and (a, b) is uniquely determined by (p, q) because of $0 \leq a < p$. Next, since each component of ∂H_1 is a torus knot in ∂W_1, we can denote it by $T(s, t)$ for some coprime integers (s, t). Then, since $T(s, t)$ intersects $T(a, b)$ in exactly two points with the same directions, we have $at - bs = \pm 2$, where one of the two points is in c_1 and the other is in c_2.

Suppose $t \neq 0$. Then $T(s, t)$ is not a longitude of W_1 and is not a meridian of W_2. Thus each component of H_2 is a ∂-parallel incompressible annulus in W_2. Let D_3 be a meridian disk of W_2 such that $\partial D_3 \cap (V_2 \cup (D_2 \times I)) = \emptyset$ and ∂D_3 is a longitude of V_1. Then by some ambient isotopy if necessary, we may assume that $H_2 \cap D_3 (\neq \emptyset)$ consists of arcs properly embedded in D_3 each of which is essential in H_2. Let α be an outermost arc component of $D_3 \cap H_2$ in D_3, Δ the outermost disk, and let A be the component of H_2 containing α. Put $\beta = cl(\partial\Delta - \alpha)$. Since β connects different components of ∂H_1, and since ∂H_1 consists of n loops corresponding to n points $\partial\tilde{E} \cap c_1$, by retaking Δ in the solid torus in W_2 cut off by A if necessary, we may assume that $\beta \subset \partial V_1 - \partial(V_2 \cup (D_2 \times I))$ and $\partial\beta \subset \partial\tilde{E}$. If $\beta \cap K = \emptyset$, then by boundary compression along Δ, we get a 2-gon in $\partial F_1 \cup ((\partial D_1^0 \cup \partial D_1^1) - \{x_1, x_2, y_1, y_2\})$, and have a contradiction by the same argument as the proof of Lemma 2.3 (4). If $\beta \cap K \neq \emptyset$, then, by taking an arc α' in the component of $\tilde{E}$ with $\partial\alpha' = \partial\beta$, the loop $\alpha \cup \alpha'$ in F bounds a disk intersecting K in a single point. Then, by Lemma 2.1 and the argument in the proof of Lemma 2.4(1) Case (1), we get a contradiction. This shows that the case when $t \neq 0$ does not occur.

Next, suppose $t = 0$. Then take the dual presentation of $T(p, q)$. Namely, take $T(q, p)$ instead of $T(p, q)$. Then, since the exterior of $T(q, p)$ is homeomorphic to the exterior of $T(p, q)$, we can perform the same arguments for the surface F as those in section 2, and we have the following two cases : (i) $F_1 = \tilde{E} \cup \tilde{A}$ and (ii) $F_1 = \tilde{E} \cup \tilde{G}$. Suppose we are in case (i). Then by performing the same arguments as those in Case I, we get the torus knot $T(b, a)$ from $T(q, p)$ because (a, b) is uniquely determined by (p, q). Then, as the next step, we get the torus knot $T(t', s')$ from $T(b, a)$ with $bs' - at' = \pm 2$. Since we are in the case when $t = 0$, we have $s = \pm 1$ and $b = \pm 2$ because of $at - bs = \pm 2$. If $s' = 0$, then $t' = \pm 1$ and $a = \pm 2$. This contradicts $\gcd(a, b) = 1$. Hence $s' \neq 0$ and we can get a contradiction similarly to the case when $t \neq 0$. If we are in case (ii), then this case is treated in th following Case II. Thus Case I does not occur except for the final case which is remained in Case II.

Suppose we are in Case II. If $\tilde{G} = \emptyset$, then $F_1 \cap \gamma_1 = \emptyset$ and $F_2 = \emptyset$. Hence $\tilde{E} = \emptyset$ and $F_1 = \emptyset$, a contradiction. Thus $\tilde{G} \neq \emptyset$ and we can put $\tilde{E} = E_1 \cup E_2 \cup \cdots \cup E_e$ and $\tilde{G} = G_1 \cup G_2 \cup \cdots \cup G_g$, where $n = e + \#(\tilde{G} \cap D_1^0)$. Note that $n = \#(D_1^0 \cap F)$. We divide this case into the following two subcases : II(1) $\tilde{G} \cap (D_1^0 \cup D_1^1) = \emptyset$ and II(2) $\tilde{G} \cap (D_1^0 \cup D_1^1) \neq \emptyset$.

Suppose we are in case II(1). Recall the notations in Case I, i.e., $W_1 = (V_1 \cup V_2) \cup (D_2 \times I)$, $W_2 = cl(S^3 - W_1) = cl(V_3 - (D_2 \times I))$, $H_1 = F \cap W_1$ and $H_2 = F \cap W_2$. In addition, we can put $\tilde{E} = E_1 \cup E_2 \cup \cdots \cup E_n$ because of $\#(\tilde{G} \cap D_1^0) = \#(\tilde{G} \cap D_1^1) = 0$. The schematic picture of $\tilde{G}, \tilde{E}, D_1^0, D_1^1, k_1, k_2$ and γ_1 is illutrated in Figure 15.

Since each component of ∂H_1 is a torus knot on ∂W_1 in S^3, we detect the torus knot type similarly to the proof of Case I. Suppose γ_1 intersects G_1 in m points. Then, since $G_1, G_2, \cdots, G_g$ are all mutually parallel meridian disks, we have $\#(\gamma_1 \cap \tilde{G}) = gm$ and $n(2r - 1) = gm$. Figure 16 illustrates the local intersection of $F_1 \cup F_2$ and $D_2 \cup \partial(V_1 \cup V_2)$ in the case when $n = 2$ and $r = 2$, and Figure 17 shows the bird's eye picture of $\partial H_1 \cap \partial W_1$ from the higher position above V_2.

Figure 17(1) is the case when $n = 2,\ r = 2,\ g = 6,\ m = 1$ and ∂H_1 consists of two

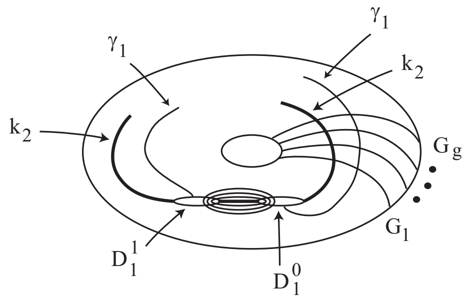


Figure 15

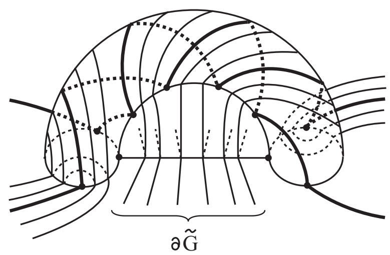


Figure 16

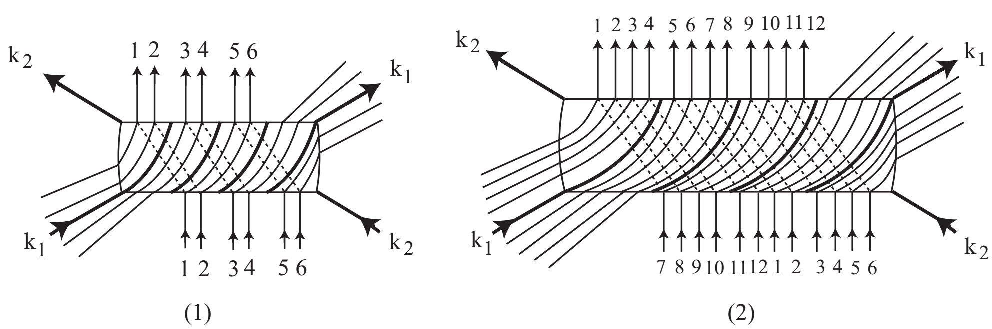


Figure 17

loops. Figure 17(2) is the case when $n = 4,\ r = 2,\ g = 6,\ m = 2$ and ∂H_1 consists of two $(= \frac{n}{2})$ loops. In general cases, by dividing the n arcs corresponding to the n points $\#(\partial F_1 \cap c_1)$ into several bunchies, we see that ∂H_1 consists of ℓ loops for some divisor ℓ of n.

Take a meridian-longitude loop pair in ∂V_1 away from $D_1^0 \cup \gamma_1 \cup D_1^1$ as illustrated in Figure 18. Count the algebraic intersection number of k_1 and the meridian-longitude pair, and put the number (p_1, q_1).

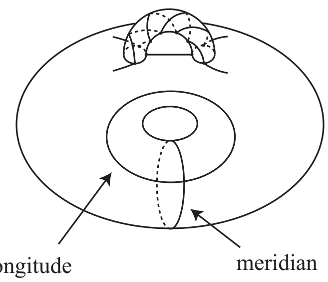


Figure 18

Then, since the algebraic intersection number of $\tilde{G}$ and the meridian-longitude pair is $(0, g)$, the algebraic intersection number of ∂H_1 and the meridian-logitude pair is $(2np_1, 2nq_1 + g)$ because the $2n$ arcs $\partial \tilde{E} - (D_1^0 \cup D_1^1)$ are parallel to k_1 (in the same direction as that of k_1) as illustrated in Figure 17. Since ∂H_1 consists of ℓ loops, the intersection number of a single loop component and the meridian-longitude pair is $(\frac{2np_1}{\ell}, \frac{1}{\ell}(2nq_1 + g))$. Suppose $\frac{1}{\ell}(2nq_1 + g) \neq 0$. Then each component of ∂H_1 is not a longitude of W_1, and is not a meridian of W_2. Thus each component of H_2 is a ∂-parallel incompressible annulus in W_2. Then by the arguments similar to the proof of the case when $t \neq 0$ in Case I, we can get a contradiction. Next suppose $\frac{1}{\ell}(2nq_1 + g) = 0$. Then $g = -2nq_1$, and by $gm = n(2r - 1)$, we have $-2nq_1 m = n(2r - 1)$, i.e., $-2q_1 m = 2r - 1$. This is a contradiction, and shows that case II(1) does not occur.

Suppose we are in case II(2). Let D_3 be a disk properly embedded in $cl(S^3 - V_1)$ such that ∂D_3 is a longitude of V_1 with $\partial D_3 \cap (V_2 \cup D_2) = \emptyset$. Since $\tilde{G} \neq \emptyset$ and each component of $\tilde{G}$ is a meridian disk of V_1, we have $\tilde{G} \cap D_3 \neq \emptyset$ and hence $D_3 \cap F_3 \neq \emptyset$, where $F_3 = cl(F - (F_1 \cup F_2))$. Then we may assume that each component of $D_3 \cap F_3$ is an arc properly embedded in D_3. Let α be an outermost arc component of $D_3 \cap F_3$ in D_3, Δ the outermost disk in D_3, and put $\beta = cl(\partial \Delta - \alpha)$. Then β is an arc in ∂V_1 with $\beta \cap F_1 = \partial \beta$ and $\beta \cap \gamma_1 = \emptyset$. Moreover we may assume that there is no 2-gon in $\beta \cup K$ which bounds a disk in ∂V_1.

Lemma 3.1 *There is an outermost arc component α of $D_3 \cap F_3$ such that there is* *no 2-gon in $\beta \cup \partial F_1$ which bounds a disk disjoint from K in ∂V_1 and β satisfies one of the following* : (1) $\beta \cap K = \emptyset$, (2) $\partial\beta \subset \partial\tilde{E}$ *or* (3) $\partial\beta \subset \partial\tilde{G}$.

Proof. If there is a 2-gon in $\beta \cup \partial F_1$ which bounds a disk disjoint from K in ∂V_1, then we can retake α and β because of $\partial D_3 \cap (V_2 \cup D_2) = \emptyset$. Hence we can assume that there is no such a 2-gon. If $\tilde{E} = \emptyset$, then $\partial\beta \subset \partial\tilde{G}$ and there is nothing to prove. Hence, we assume $\tilde{E} \neq \emptyset$. First we note that $\#(\tilde{G} \cap D_1^0) = \#(\tilde{G} \cap D_1^1)$ because $\#(F_1 \cap D_1^0) = \#(F_1 \cap D_1^1) = n$ and $\#(\tilde{E} \cap D_1^0) = \#(\tilde{E} \cap D_1^1)$. Then we may assume that E_1 is the outermost component of $\tilde{E}$ and that G_1 is the component of $\tilde{G}$ with $G_1 \cap (D_1^0 \cup D_1^1) \neq \emptyset$ such that G_1 is the closest component to E_1 in D_1^0 as in Figure 19.

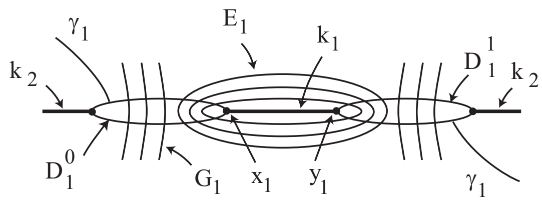


Figure 19

Let α be an outermost arc component of $D_3 \cap F_3$, and β the corresponding arc in ∂D_3. Suppose $\beta \cap K \neq \emptyset$ and β connects a component of $\partial\tilde{E}$ and a component of $\partial\tilde{G}$. Then we have the following two subcases : case (a) β connects E_1 and the component G_1, case (b) β connects E_1 and a component G_2 of $\tilde{G}$ different from G_1.

Suppose we are in case (a). First suppose β meets G_1 from the same side as the component $D_1^0 \cap E_1$. Then we have the situations as in Figure 20(1). If there are intersections of β and k_2 as in Figure 20(2), then we can remove the intersection by sliding as in Figure 20(3), where such a sliding corresponds to an ambient isotopy to retake the disk D_3. Thus, the case when we need to consider is the situation as in Figure 21(1), i.e., the case when there is an intersection of D_1^1 and ∂G_1 which obstructs the sliding as in Figure 20(3). Then, since $\gamma_1 \cap \beta = \emptyset$ and $\gamma_1 \cap K = \emptyset$, we have a 3-gon as illustrated in Figure 21(2). This is a contradiction by Lemma 2.5(2). Next suppose β meets G_1 from the opposite side as the component $D_1^0 \cap E_1$ as in Figure 22. Then $\tilde{G} = G_1$ and this means F is a non-separating closed surface in S^3, a contradiction. Thus case (a) does not occur.

Next suppose we are in case (b). First we note that G_1 and G_2 are mutually parallel meridian disks and β is contained in the region between G_1 and G_2. Then we have the following two subcases : case (b-1) G_2 is the closest component to E_1 in D_1^1 as in Figure 23(1), case (b-2) G_1 intersects D_1^1 too and is the closest component to E_1 in

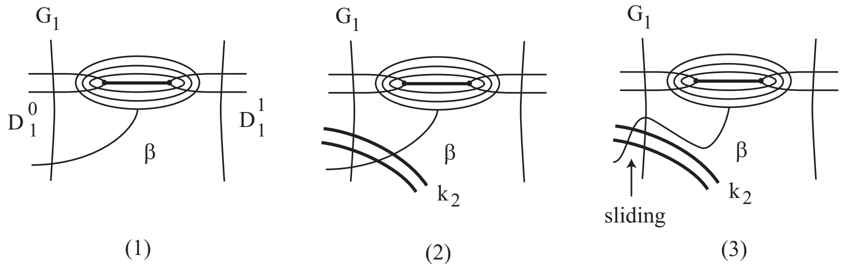


Figure 20

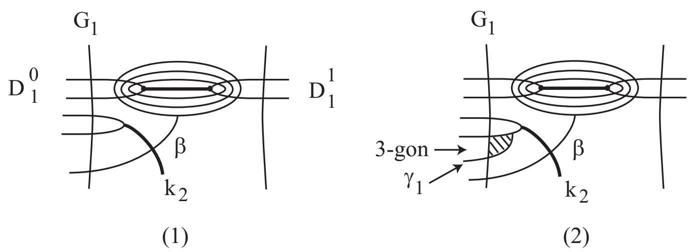


Figure 21

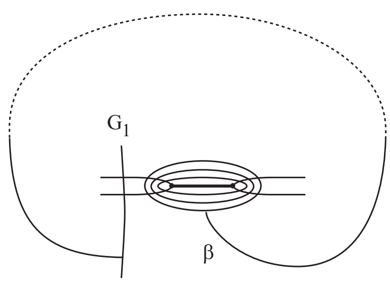


Figure 22

D_1^1 as in Figure 23(2).

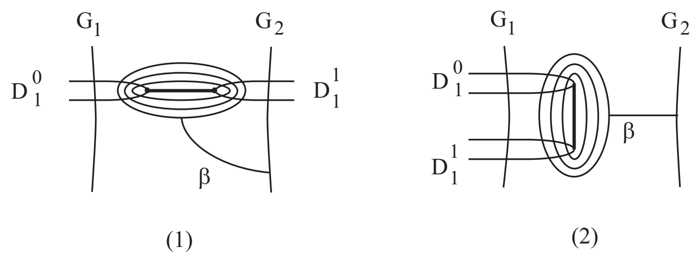


Figure 23

Suppose we are in case (b-1). Then, since β connects E_1 and G_2, this case is the same one as in case (a), and we have a contradiction. Next suppose we are in case (b-2). Then, β can be slided to remove the intersection $\beta \cap K$ as in Figure 20, or by noting that $\#(\tilde{G} \cap D_1^0) = \#(\tilde{G} \cap D_1^1)$ we have the situation as in Figure 24, which is the case when there are intersections of G_2 and $D_1^0 \cup D_1^1$ which obstrcuts the sliding. Then, there is a 2-gon in $\partial\tilde{G} \cup k_2$ as in Figure 24(1), which contradicts Lemma 2.3(2), or there is a 3-gon which consists of a subarc of $\partial D_1^0 \cup \partial D_1^1$, a subarc of ∂G_2 and a subarc of k_2 as in Figure 24(2), which contradicts Lemma 2.5(3). Hence $k_2 \cap \tilde{G} = \emptyset$ as in Figure 24(3). Then, there is a 3-gon which consists of a subarc of $\partial D_1^0 \cup \partial D_1^1$, a subarc of ∂G_2 and a subarc of γ_1 as in Figure 24(3), which contradicts Lemma 2.5(2). Hence case (b) does not occur, and this completes the proof of Lemma 3.1. □

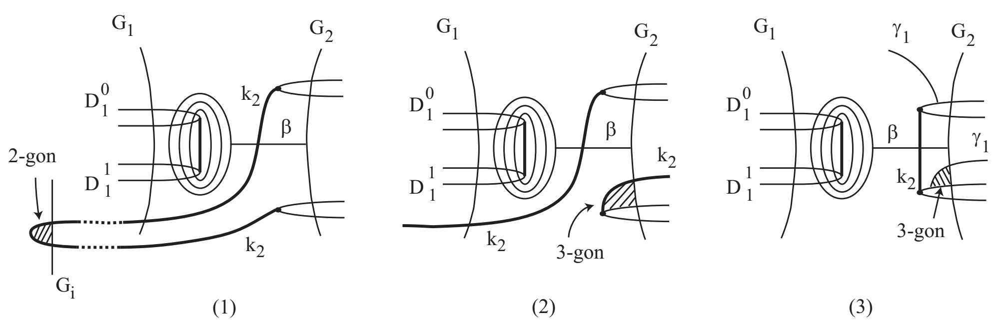


Figure 24

Now, let α be an outermost arc component of $F \cap D_3$ in D_3, Δ the outermost disk and $\beta = cl(\partial\Delta - \alpha)$. Then, by Lemma 3.1, we may assume that β satisfies one of the following three cases. Case II(2)-a : $\beta \cap K = \emptyset$ and β connects a component of $\partial\tilde{E}$ and a component of $\partial\tilde{G}$, Case II(2)-b : $\partial\beta \subset \partial\tilde{E}$, Case II(2)-c : $\partial\beta \subset \partial\tilde{G}$. Hereafter we assume that $\#(F_1)$ is minimal among all essential surfases F satisfying the condition of Case II.

Suppose we are in Case II(2)-a. In this case $\beta \cap K = \emptyset$ and β connects a component of $\tilde{E}$, say E_1, and a component of $\tilde{G}$, say G_1. Let b be a band in V_1 produced by a boundary compression along Δ, and put $G_0 = E_1 \cup b \cup G_1$. Then G_0 is a meridian disk properly embedded in V_1 and G_0 can be regarded as a component of $\tilde{G}$. This means that $\#(\tilde{E})$ decreas by one and $\#(\tilde{G})$ does not change, and contradicts the minimality of $\#(F_1)$. Thus Case II(2)-a does not occur.

Suppose we are in Case II(2)-b. In this case, since $\beta \cap \tilde{G} = \emptyset$, $\beta \cap \gamma_1 = \emptyset$ and $\beta \cap (D_1^0 \cup D_1^1) = \emptyset$, we have the three cases : ① $\partial\beta$ connects different components of $\partial\tilde{E}$, ② β meets the innermost component of $\partial\tilde{E}$, ③ β meets the outermost component of $\partial\tilde{E}$ as illustareted in Figure 25.

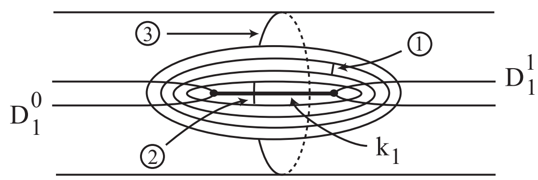


Figure 25

In case ①, by a boundary compression along Δ, we get a 2-gon in $\partial F_1 \cup ((\partial D_1^0 \cup \partial D_1^1) - \{x_1, x_2, y_1, y_2\})$ and we can decrease the number n, a contradiction. In case ②, we can get a disk as in Lemma 2.1, a contradiction. In case ③, let b be the band in V_1 produced by a boundary compression along Δ. Then, since $b \cap \tilde{G} = \emptyset$, $b \cup E_1$ is a compressible annulus in V_1, and let D be the compressing disk. Then, by the incompressibility of $F - K$ in $S^3 - K$, ∂D bounds a disk in $F - K$, say D'. Then by an ambient isotopy through the 3-ball bounded by $D \cup D'$, we can remove somoe components of $F \cap V_2 = F_2$. This contradicts the minimality of $n = \#(F_2) = \#(F_1 \cap D_1^0$. Hence Case II(2)-b does not occur.

Suppose we are in Case II(2)-c. First suppose $\beta \cap K = \emptyset$, and let b be the band in V_1 produced by a boundary compression along Δ. Suppose b meets a single component of $\tilde{G}$, say G_1. If b meets G_1 from both sides, then $b \cup G_1$ is a Möbius band, a contradiction. If b meets G_1 from one side, then there is a 2-gon in $\beta \cup \partial G_1$ which bounds a disk disjoint from K in ∂V_1, a contradiction. Hence we may assume that b connects two components G_1 and G_2. Then, since G_1 and G_2 are mutually parallel meridian disks of V_1, we have the following four cases related to the neighborhood of b : (1) b is parallel to k_2, (2) b is parallel to D_1^0 or to D_1^1, (3) b is parallel to $D_1^0 \cup k_2$ or to $D_1^1 \cup k_2$, (4) b is parallel to $D_1^0 \cup k_1 \cup D_1^1$ as illustrated in Figure 26.

Put $E_0 = G_1 \cup b \cup G_2$. Then E_0 is a ∂-parallel disk In cases (1), (2), we have a contradiction by Lemma 2.3. In case (3), we have a contradiction by Lemma 2.4 (Figure 6). In case (4), if $(\partial G_1 \cup \partial G_2)$ intersects k_2 as in the cases (1) or (3), or

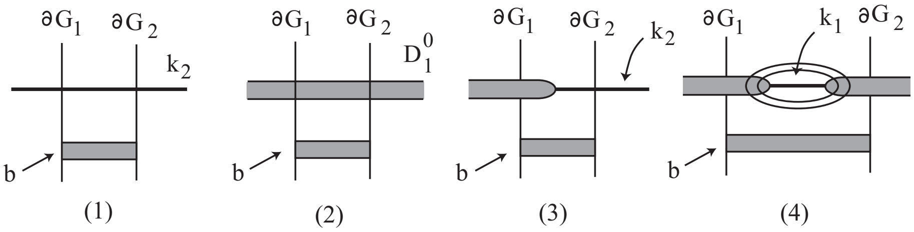


Figure 26

intersects $D_1^0 \cup D_1^1$ as in case (2), then we have a contradiction similarly. Hence $(\partial G_1 \cup \partial G_2) \cap k_2 = \emptyset$ and $(\partial G_1 \cup \partial G_2) \cap (D_1^0 \cup D_1^1)$ consists of exactly two arcs as in Figure 26(4). Then E_0 can be regarded as a component of $\tilde{E}$, and this means that $\#(\tilde{E})$ increase by one and $\#(\tilde{G})$ decrease by two. This contradicts the minimality of $\#(F_1)$, and case (4) does not occur.

Next suppose $\beta \cap K \neq \emptyset$. If we can remove the intersection by some sliding as in Figure 20, then this case can be regarded as the case when $\beta \cap K = \emptyset$. Hence we assume that $\beta \cap K$ can not be removed by sliding. Put $X = \{x_1, x_2, y_1, y_2\}$.

First suppose β meets a single component of $\tilde{G}$, say G_1. Then the disk bounded by the 2-gon $\beta\cup$ (a subarc of ∂G_1) contains some points of X. If it contains three or four points of X, then, by considering the intersections of $D_1^0 \cup D_1^1$ and $\partial\tilde{G}$, we can find a 2-gon in $\partial\tilde{G} \cup ((\partial D_1^0 \cup \partial D_1^1) - X)$, and this contradicts Lemma 2.3(4). Hence it contains one or two point(s) of X as in Figure 27. If it contains one point then we have a contradiction by Lemma 2.4 (Figure 6). Suppose it contains two points. If one of the two points is x_2 or y_2, then we can find a 3-gon as in Lemma 2.5(2) (c.f. Figure 21(2)), a contradiction. Thus the two points are x_1 and y_1. Such a case can occur in the case when $\tilde{E} = \emptyset$. Then, since k_1 connects x_1 and y_1, we can find a 2-gon in $k_1 \cup \tilde{G}$ which bounds a disk in ∂V_1 (c.f. Figure 24(1)). This contradicts Lemma 2.3(2).

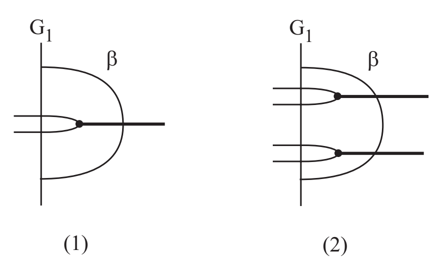


Figure 27

Next suppose β connects two different components of $\tilde{G}$, say G_1 and G_2, and let R be the region (an annulus) in ∂V_1 between G_1 and G_2 containing β. If R contains at most one point of X, then we can remove intersections of β and K by sliging. Hence $R \cap X$ consists of two, three or four points.

Suppose $\#(R \cap X) = 2$. Then we have the three situations illustrated in Figure 28, which are the case when $\beta \cap K$ cannot be removed by sliding. Suppose we are in the situation of Figure 28(1). If at least one of the two points of X in R is x_2 or y_2, then we can find a 3-gon as in Lemma 2.5(2), a contradiction. Hence the two points are x_1 and y_1. Then, since there are at lease two outermost components of $F_3 \cap D_3$, take another outermost component, say α', and let β' be the corresponding arc in ∂D_3. If β' is contained in R, then, since β and β' are mutually parallel arcs in R, and since ∂D_3 is a longitude of V_1, we can find a 2-gon in $\partial D_3 \cup \partial \tilde{G}$ which bounds a disk in ∂V_1. Then the disk contains x_2 or y_2 and we can find a 3-gon as in Lemma 2.5(2), a contradiction. If β' is contained in the region different from R, say R', then we can remove the intersection of β' and K, or the same situations as in Figure 28 happen. In the former case, we have a contradiction similarly to the case wehen $\beta \cap K = \emptyset$. In the latter case, we can find a 3-gon as in Lemma 2.5(2) or (3) because $R' \cap X = \{x_2, y_2\}$, and this is a contradiction.

Finally, consider the case when $\#(R \cap X) = 3$ or 4. Then by the arguments similar to the above, we can find a 3-gon as in Lemma 2.5(2) or (3), a contradiction. Thus Case II(2)-c does not occur, and hence Case II does not occur. After all, either Case I nor Case II occurs, and we complete the proof of Theorem 1.1. □

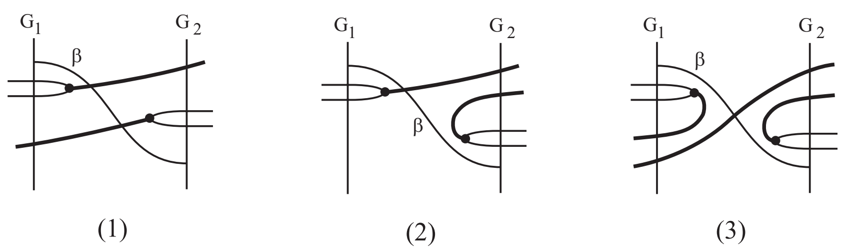


Figure 28

References

[CGLS] M. Culler, C. McA. Gordon, J. Lueke and P. B. Shalen, *Dehn surgery on knots*, Ann. of Math. **115** (1987) 237-300

[Ho] P. Hoiden, *On 1-bridge genus of small knots*, Topology Appl. **106** (2000) 321-335

[M1] K. Morimoto, *On the supper additivity of tunnel number of knots*, Math. Ann. **317** (2000) 489-508

[M2] K. Morimoto, *Characteization of composite knots with 1-bridge genus two*,

J. K. T. R. **10** (2001) 823-840
[M3] K. Morimoto, *Tunnel number,* 1*-bridge genus and h-genus of knots*, Topology Appl. **146** (2005) 149-158
[M4] K. Morimoto, *A note on self-fusion of torus knots*, Memoirs of Konan Unversity Science and Engineering Series, **53** (2006) 1-8
[MSY] K. Morimoto, M. Sakuma and Y. Yokota, *Examples of tunnel number one knots which have the property* " $1 + 1 = 3$ ", Math. Proc. Camb. Phil. Soc. **119** (1996) 113-118
[MY] K. Morimoto and Y. Yamada, *A note on essential tori in the exteriors of torus knots with twists*, Preprint
[No] F. Norwood, *Every two generator knot is prime*, Proc. A. M. S. **86** (1982) 143-147
[Ro] D. Rolfsen, *Knots and Links*, AMS Chelsea Publishing (1976, 2003)
[Sc] M. Scharlemann, *Tunnel number one knots satisfy the Poenaru conjecture*, Topology Appl. **18** (1984) 235-258