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ON TANGLE DECOMPOSITIONS OF TWISTED TORUS KNOTS

KANJI MORIMOTO

Department of IS and Mathematics,
Konan University, Okamoto 8-9-1,
Higashi-Nada, Kobe 658-8501, Japan
morimoto@konan-u.ac.jp

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ABSTRACT

In the present paper, we will show that for any integer $n > 0$ there are infinitely many twisted torus knots with n-string essential tangle decompositions, and that all those knots have essential tori in the exteriors.

Keywords: Twisted torus knots; tangle decompositions.

Mathematics Subject Classification 2010: 57M25, 57M27

1. Introduction

Let p, q, r, s be integers with $p > r > 1$, $q > 0$, $\gcd(p, q) = 1$, and let $T(p, q)$ be the torus knot of type (p, q) in S^3. For the definition of torus knots $T(p, q)$ we refer to [10]. Add s times full twists on mutually parallel r-strands in $T(p, q)$. Then according as [2], we call the knot obtained by this operation a twisted torus knot of type $(p, q; r, s)$ and denote it by $T(p, q; r, s)$ as illustrated in Fig. 1.

Recently, several interesting results on twisted torus knots have been gotten [3–9]. In particular, we have shown in [6] that there are infinitely many composite twisted torus knots as follows.

Theorem 1.1 ([6, Theorem 1]). *Let $e > 0, k_1 > 1, k_2 > 1$ be integers, and put*

$$p_0 = (e+1)(k_1+k_2)+1, \quad q_0 = e(k_1+k_2)+1,$$
$$r_0 = p_0 - k_1 \quad \text{and} \quad s_0 = -1.$$

Then $T(p_0, q_0; r_0, s_0)$ is the connected sum of the two torus knots $T(k_1, ek_1+1)$ and $T(k_2, -(e+1)k_2-1)$.

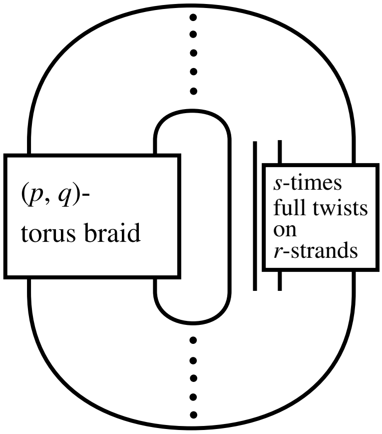

Fig. 1. $T(p, q; r, s)$.

In the present paper, as an extension of the above result, we will show that there are infinitely many twisted torus knots with n-string essential tangle decompositions for any integer $n > 0$, and that those all knots have essential tori in the exteriors as follows.

Theorem 1.2. *Let* $e > 0, k_1 > 1, k_2 > 1, x_1 > 0, x_2 > 0$ *be integers with* $\gcd(x_1, x_2) = 1$. *Put*

$$p = ((e+1)(k_1+k_2-1)+1)x_1 + (e+1)x_2,$$

$$q = (e(k_1+k_2-1)+1)x_1 + ex_2,$$

$$r = ((e+1)(k_1+k_2-1)-k_1+2)x_1 + ex_2 \quad \textit{and} \quad s = -1.$$

Then we have the following:

(1) $T(p, q; r, s)$ *has an* x_1*-string essential tangle decomposition.*
(2) *The decomposition is obtained by the* x_1*-string fusion of the torus knot* $T((k_1-1)x_1+x_2, e((k_1-1)x_1+x_2)+x_1)$ *and the torus link* $T(k_2x_1, -((e+1)k_2+1)x_1)$.
(3) $T(p, q; r, s)$ *has an essential torus in the exterior whose companion is the torus knot* $T(k_2, -(e+1)k_2-1)$.

Therefore, for any integer $n > 0$, *by putting* $x_1 = n$ *we get infinitely many twisted torus knots with* n*-string essential tangle decompositions.*

Example 1.3. Put $e = 1, k_1 = k_2 = 2, x_1 = 2, x_2 = 3$. Then by Theorem 1.2, we see that $T(20, 11; 15, -1)$ has a 2-string essential tangle decomposition which is obtained by the 2-string fusion of $T(5, 7)$ and $T(4, -10)$ as in Fig. 2 (cf. Example 3.2). In addition, by connecting the decomposing 2-sphere with a tube along the torus link $T(4, -10)$, we have an essential torus whose companion is the torus knot $T(2, -5)$.

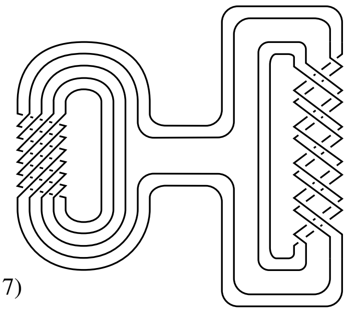

Fig. 2. $T(20,11:15,-1)$

Remark 1.4. Suppose $x_1 = 1$ in Theorem 1.2. Then by putting $k_1' = k_1 + x_2 - 1$ and $k_2' = k_2$, we have:

$$p = (e+1)(k_1 + k_2 - 1) + 1 + (e+1)x_2 = (e+1)(k_1' + k_2') + 1,$$
$$q = e(k_1 + k_2 - 1) + 1 + ex_2 = e(k_1' + k_2') + 1,$$
$$r = (e+1)(k_1 + k_2 - 1) - k_1 + 2 + ex_2 = p - k_1' \quad \text{and} \quad s = -1.$$

This shows that $T(p, q; r, s)$ is a composite twisted torus knot if $x_1 = 1$ as in Theorem 1.1.

Concerning essential tori in the exteriors of twisted torus knots, Lee showed the following theorem (cf. [9]).

Theorem 1.5 ([4, Theorem 1]). *Suppose* $r \equiv 0 \pmod q$. *Then by putting* $r = qk$ *for some integer* k, $T(p, q; r, s)$ *is the* $(q, p + k^2qs)$*-cable knot along the torus knot* $T(k, ks + 1)$.

Hence we can ask the following question.

Question. *Are there twisted torus knots with essential tori which are not in Theorem* 1.2 *or in Theorem* 1.5?

Concerning the above question, Lee has been recently shown the following theorem.

Theorem 1.6 ([5, Theorem 1]). *Suppose* $r \not\equiv 0 \pmod q$ *and* $T(p, q; r, s)$ *contains an essential torus in the exterior. Then* $|s| \leq 2$.

Remark 1.7. Concerning the problem on the existence of essential closed surfaces (not essential tori) in the exteriors of twisted torus knots, Theorem 1.2 says nothing at all. Because the closed surfaces obtained by connecting the decomposing 2-spheres with a tube along the strings of the tangles are not essential surfaces. On the essential surfaces in the exteriors of twisted torus knots, it has been shown in [7] that $T(p, q; r, s)$ has no closed essential surfaces if $r = 2$.

Throughout the present paper, we will work in the piecewise linear category. For a manifold X and a subcomplex Y in X, we denote a regular neighborhood of Y in X by $N(Y, X)$ or $N(Y)$ simply.

2. Parallelized Torus Knots and Parallelized Twisted Torus Knots

Let $T(p_0, q_0)$ be the torus knot of type (p_0, q_0), where p_0 and q_0 are positive coprime integers with $p_0 > 1$, and let x_1 and x_2 be positive integers. Take four points P_1, P_2, P_3 and P_4 on the adjacent two strands in $T(p_0, q_0)$ as in Fig. 3. Then replace the arc P_1 through P_3 with x_1 parallel strings and the arc P_2 through P_4 with x_2 parallel strings. In addition, replace the rectangle $P_1P_2P_3P_4$ with $x_1 + x_2$ strands as in Fig. 4. Then we get a torus knot or a torus link $T(p, q)$ for some p, q.

Let us detect p and q. First, number the p_0 strings below the (p_0, q_0)-torus braid $0, 1, 2, \ldots, p_0 - 2, p_0 - 1$ as in Fig. 5. The arc starting at P_1 goes into the braid at $p_0 - 1$ and goes out at $q_0 - 1$. After round once, it goes into the braid again and goes out at $2q_0 - 1$. Next it goes out the braid at $3q_0 - 1$. By continuing these procedures, it finally goes out at $aq_0 - 1 \equiv p_0 - 2 \pmod{p_0}$ for some a. Then it meets the point P_3. Hence we have $aq_0 \equiv -1 \pmod{p_0}$. Similarly the arc starting

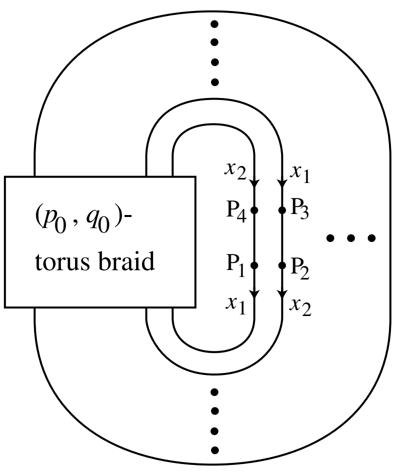

Fig. 3. $T(p_0, q_0)$.

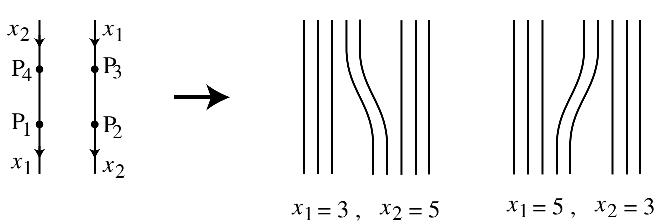

Fig. 4. Parallelization.

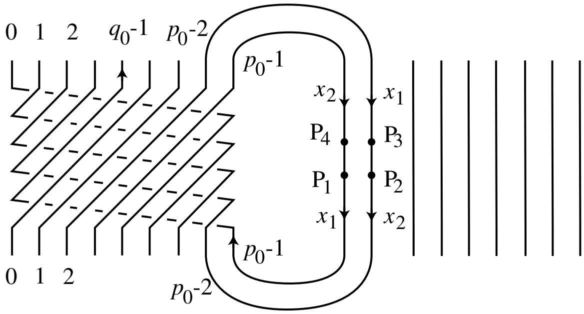

Fig. 5. Detecting p and q.

at P_2 goes into the braid at $p_0 - 2$ and goes out at $q_0 - 2$. Then it goes out the braid at $2q_0 - 2, 3q_0 - 2, \ldots$, and finally goes out at $bq_0 - 2 \equiv p_0 - 1 \pmod{p_0}$ for some b. Then it meets the point P_4. Hence we have $bq_0 \equiv 1 \pmod{p_0}$.

Thus we have $p = ax_1 + bx_2$, where a and b are the least positive integers such that $aq_0 \equiv -1 \pmod{p_0}$, $bq_0 \equiv 1 \pmod{p_0}$ and $a + b = p_0$.

By the similar arguments, we have $q = cx_1 + dx_2$, where c and d are the least positive integers such that $cp_0 \equiv 1 \pmod{q_0}$, $dp_0 \equiv -1 \pmod{q_0}$ and $c + d = q_0$.

In general, we have the following proposition.

Proposition 2.1. *For coprime positive integers p_0 and q_0, there uniquely exist positive integers a, b, c, d which satisfy the following conditions*:

$$(1)\ \begin{cases} a + b = p_0, \\ aq_0 \equiv -1 \pmod{p_0}, \\ bq_0 \equiv 1 \pmod{p_0}, \end{cases} \quad (2)\ \begin{cases} c + d = q_0, \\ cp_0 \equiv 1 \pmod{q_0}, \\ dp_0 \equiv -1 \pmod{q_0}. \end{cases}$$

Proof. Consider the set $\{0, q_0, 2q_0, \ldots, (p_0 - 1)q_0\}$. Then, by $\gcd(p_0, q_0) = 1$, these p_0 integers are different to each other (mod p_0). Then this set coincides with the set $\{0, 1, 2, \ldots, p_0 - 1\}$ (mod p_0), and hence there is only one integer a with $aq_0 \equiv p_0 - 1 \equiv -1 \pmod{p_0}$. Then by putting $b = p_0 - a$, we have $bq_0 = (p_0 - a)q_0 = p_0q_0 - aq_0 \equiv 0 - (-1) = 1 \pmod{p_0}$. This completes the proof of (1). The condition (2) is proved similarly. □

Under the above situations, we have the following proposition.

Proposition 2.2. *Let x_1 and x_2 be positive integers, and put $p = ax_1 + bx_2$ and $q = cx_1 + dx_2$. Then $\gcd(p, q) = \gcd(x_1, x_2)$. In particular, $T(p, q)$ is a torus knot if and only if $\gcd(x_1, x_2) = 1$.*

Proof. Put $\gcd(x_1, x_2) = k$. Then we can put $x_1 = ky_1$, $x_2 = ky_2$ for some y_1, y_2, and put $p = k(ay_1 + by_2)$, $q = k(cy_1 + dy_2)$. Hence $\gcd(p, q) \geq k = \gcd(x_1, x_2)$.

Conversely, put $\gcd(p, q) = k$. Then we can put $p = kp_1$, $q = kq_1$ for some p_1, q_1.

Since $\begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, we have

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} kp_1 \\ kq_1 \end{bmatrix}. \tag{1}$$

Then $|ad - bc| < p_0 q_0 - 1$ because $0 < a, b < p_0$ and $0 < c, d < q_0$. Moreover $ad - bc = a(q_0 - c) - (p_0 - a)c = aq_0 - cp_0 \equiv -1 \pmod{p_0}, \pmod{q_0}$. This implies that $ad - bc = -1$, and by Eq. (1) we have $\gcd(x_1, x_2) \geq k = \gcd(p, q)$. This completes the proof. □

By summarizing the above arguments, we have the following proposition.

Proposition 2.3. *Let $T(p_0, q_0)$ be the torus knot of type (p_0, q_0) with $p_0 > 1$, $q_0 > 0$, $\gcd(p_0, q_0) = 1$, and let x_1, x_2 be positive integers. Then by the parallelization of $T(p_0, q_0)$, we have a torus knot or a link $T(p, q)$ with $p = ax_1 + bx_2$ and $q = cx_1 + dx_2$, where (a, b, c, d) are uniquely determined by the conditions in Proposition* 2.1.

Next, let $T(p_0, q_0; r_0, s_0)$ be a twisted torus knot. Then by the same way as the case of torus knots, we can construct a parallelized twisted torus knot or a link $T(p, q; r, s)$. Then $p = ax_1 + bx_2, q = cx_1 + dx_2, r = r_1x_1 + r_2x_2$ and $s = s_0$, where a, b, c, d are those integers in Proposition 2.1 and r_1, r_2 are some positive integers with $r_1 + r_2 = r_0$.

Let us detect r_1 and r_2. To do this, we need to count the numbers of the intersection of the arc P_1 through P_3 and the box of the r_0-strings in Fig. 6. Then, since the arc P_1 through P_3 goes out from the (p_0, q_0)-torus braid at the string $kq_0 - 1 \pmod{p_0}$ $(k = 1, 2, \ldots, a)$, by the same arguments as those to determine the integer a in Proposition 2.1, we can put r_1 and r_2 as follows, where # is the cardinal number of the given set:

$$\begin{cases} r_1 = \#\{k \mid 0 \leq kq_0 - 1 \pmod{p_0} \leq r_0 \ (k = 1, 2, \ldots, a)\}, \\ r_2 = r_0 - r_1. \end{cases} \tag{$*$}$$

By summarizing the above arguments, we have the following proposition.

Proposition 2.4. *Let $T(p_0, q_0; r_0, s_0)$ be the twisted torus knot of type $(p_0, q_0; r_0, s_0)$ with $p_0 > r_0 > 1$, $q_0 > 0$, $\gcd(p_0, q_0) = 1$, and let x_1, x_2 be positive integers. Then by the parallelization of $T(p_0, q_0; r_0, s_0)$, we have a twisted torus knot or a link $T(p, q; r, s)$ with $p = ax_1 + bx_2$, $q = cx_1 + dx_2$, $r = r_1x_1 + r_2x_2$ and $s = s_0$, where (a, b, c, d) and (r_1, r_2) are uniquely determined by the conditions in Proposition* 2.1 *and the above condition* $(*)$.

We note that $T(p, q; r, s)$ is a knot if and only if $\gcd(x_1, x_2) = 1$ by Proposition 2.2.

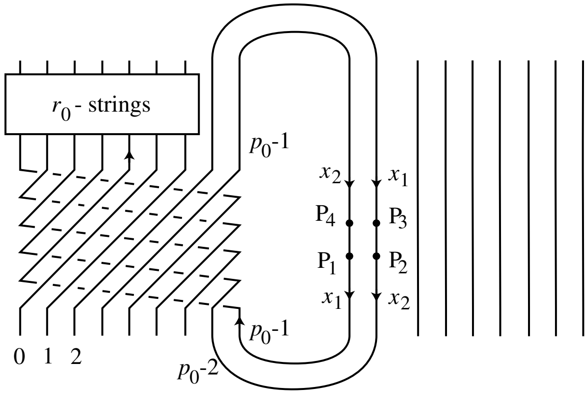

Fig. 6. Detecting r_1 and r_2.

3. Proof of Theorem 1.2

Let B be a 3-ball, and let $t^1 \cup t^2 \cup \cdots \cup t^n$ be the n arcs properly embedded in B. Then we call the pair $(B, t^1 \cup t^2 \cup \cdots \cup t^n)$ an n-string tangle. We say that $(B, t^1 \cup t^2 \cup \cdots \cup t^n)$ is essential if $\mathrm{cl}(\partial B - N(t^1 \cup t^2 \cup \cdots \cup t^n))$ is incompressible in $\mathrm{cl}(B - N(t^1 \cup t^2 \cup \cdots \cup t^n))$ if $n > 1$, and t^1 is a knotted arc in B if $n = 1$, where $N(t^1 \cup t^2 \cup \cdots \cup t^n)$ is a regular neighborhood of $t^1 \cup t^2 \cup \cdots \cup t^n$ in B, and that the tangle is inessential if it is not essential. We say that a knot K in the 3-sphere S^3 has an n-string essential tangle decomposition if (S^3, K) is decomposed into two n-string essential tangles $(B_1, t_1^1 \cup t_1^2 \cup \cdots \cup t_1^n) \cup (B_2, t_2^1 \cup t_2^2 \cup \cdots \cup t_2^n)$, and that the decomposition is inessential if it is not essential.

To prove Theorem 1.2, we construct parallelized twisted torus knots from composite twisted torus knots, and we will show that the decomposing 2-sphere of the connected sum becomes the decomposing 2-sphere of the tangle decomposition.

Recall Theorem 1.1 (see [6, Theorem 1]). To get parallelized twisted torus knots from the composite knots in Theorem 1.1, first we calculate the integers a, b, c, d in Proposition 2.1 to get p and q.

Proposition 3.1. *Put* $p_0 = (e+1)(k_1 + k_2) + 1$ *and* $q_0 = e(k_1 + k_2) + 1$. *Then those integers* a, b, c, d *in Proposition* 2.1 *are as follows*:

$$a = (e+1)(k_1 + k_2 - 1) + 1, \quad b = e + 1,$$
$$c = e(k_1 + k_2 - 1) + 1 \quad \textit{and} \quad d = e.$$

Proof. First we have $b = e + 1$, because $(e+1)q_0 = (e+1)(e(k_1 + k_2) + 1) = (e+1)e(k_1 + k_2) + e + 1 = e((e+1)(k_1 + k_2) + 1) + 1 = ep_0 + 1 \equiv 1 \pmod{p_0}$. Then $a = p_0 - b = (e+1)(k_1 + k_2) + 1 - (e+1) = (e+1)(k_1 + k_2 - 1) + 1$ and $aq_0 = (p_0 - b)q_0 = p_0 q_0 - bq_0 \equiv -1 \pmod{p_0}$.

Next we have $d = e$ because $ep_0 = e((e+1)(k_1+k_2)+1) = e(e+1)(k_1+k_2)+e = (e+1)e(k_1+k_2)+e+1-1 = (e+1)(e(k_1+k_2)+1)-1 = (e+1)q_0 - 1 \equiv -1 \pmod{q_0}$.

Then $c = q_0 - d = e(k_1 + k_2) + 1 - e = e(k_1 + k_2 - 1) + 1$ and $cp_0 = (q_0 - d)p_0 = q_0p_0 - dp_0 \equiv 1 \pmod{q_0}$. This completes the proof. □

To calculate r and to get concrete expression of the tangle decompositions, consider the composite twisted torus knots in Theorem 1.1. Put $K_0 = T(p_0, q_0; r_0, s_0)$, $K_1 = T(k_1, ek_1 + 1)$ and $K_2 = T(k_2, -(e+1)k_2 - 1)$, then $K_0 = K_1 \# K_2$ as in Theorem 1.1. Let V be a standard genus two handlebody in S^3, and put $\partial V = F$. Then, since any twisted torus knot can be embedded in F in a standard way, we may assume that K_0 is in F. Let S be the decomposing 2-sphere of the connected sum $K_0 = K_1 \# K_2$; then, by the proof of Theorem 1.1 in [6], we may assume that S intersects V in a single separating disk which is a union of two mutually parallel non-separating disks and a band, and that $S \cap F = \partial(S \cap V)$ is a single loop. Then, by noting that $p_0 - r_0 = k_1$, $S \cap F$ runs along the both sides of k_1 strings and $(S \cap F) \cap K_0$ consists of the two points Q_1 and Q_2 as indicated in Fig. 7, where Fig. 7 is the case of $e = 1, k_1 = 3, k_2 = 2$ and the connected sum is $T(11, 6; 8, -1) = T(3, 4) \# T(2, -5)$.

We split K_0 at Q_1, Q_2 into two arcs, and connect the two points with the arc in the disk $S \cap V$. Then we get the two knots K_1 and K_2. First we consider K_1 as in Fig. 8, where Fig. 8 is the case of $k_1 = 5$. Then, by noting that $p_0 - r_0 = k_1$ and

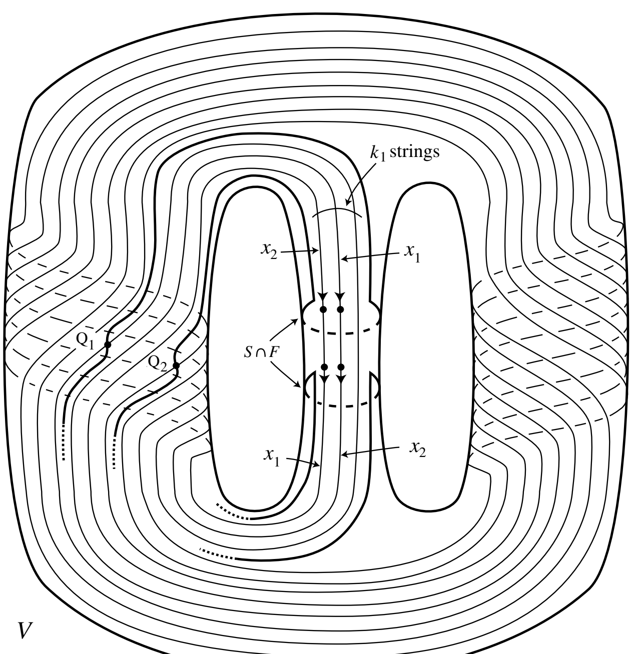

Fig. 7. K_0 and $S \cap F$ in F.

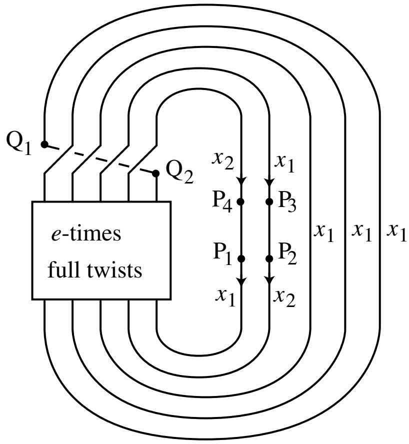

Fig. 8. Knot K_1.

$K_1 = T(k_1, ek_1 + 1)$, we see that the arc P_2 through P_4 is contained in K_1, and hence exactly one string of k_1 strings is replaced with x_2 parallel strings. Then the other (k_1-1) strings are contained in the arc P_1 through P_3 and are replaced with x_1 parallel strings. Thus we get the torus knot $T((k_1-1)x_1+x_2, e((k_1-1)x_1+x_2)+x_1)$, and this torus knot intersects the original decomposing 2-sphere in x_1 points at each $\mathrm{Q}_i (i = 1, 2)$. This implies that $r = p - ((k_1 - 1)x_1 + x_2)$.

For the knot K_2, by the above arguments, we see that the whole string of K_2 is contained in the arc P_1 through P_3. Hence by replacing the whole string with x_1 parallel strings, we get the torus link $T(k_2x_1, -((e + 1)k_2 + 1)x_1)$. This torus link intersects the original decomposing 2-sphere in x_1 points at each $\mathrm{Q}_i (i = 1, 2)$ similarly to the case of K_1.

By summarizing the above calculations, we have the following, and get the knots in Theorem 1.2:

$$
\begin{aligned}
p &= ax_1 + bx_2 = ((e + 1)(k_1 + k_2 - 1) + 1)x_1 + (e + 1)x_2,\\
q &= cx_1 + dx_2 = (e(k_1 + k_2 - 1) + 1)x_1 + ex_2,\\
r &= p - ((k_1 - 1)x_1 + x_2) = ((e + 1)(k_1 + k_2 - 1) + 1)x_1\\
&\quad + (e + 1)x_2 - ((k_1 - 1)x_1 + x_2)\\
&= ((e + 1)(k_1 + k_2 - 1) - k_1 + 2)x_1 + ex_2,\\
s &= s_0 = -1.
\end{aligned}
$$

Finally, we need to show that the above tangle decompositions are all essential. If $x_1 = 1$, then the decompositions are the connected sums and are all essential because both k_1 and k_2 are greater than one and factor knots are non-trivial knots.

Suppose $x_1 > 1$. By the definition of tangles, we see that an n-string tangle $(B, t^1 \cup t^2 \cup \cdots \cup t^n)$ with $n > 1$ is essential if and only if there is no disk properly embedded in B which separates the arcs $t^1 \cup t^2 \cup \cdots \cup t^n$. From this viewpoint, in the next section, we will show that both of x_1-string tangles constructed from the torus knot $T((k_1-1)x_1+x_2, e((k_1-1)x_1+x_2)+x_1)$ and the torus link $T(k_2x_1, -((e+1)k_2+1)x_1)$ are essential (Propositions 4.2 and 4.3).

In addition, by connecting the decomposing 2-sphere with a tube along the torus link $T(k_2x_1, -((e+1)k_2+1)x_1)$ with x_1-string bunches, we have an essential torus whose companion is the torus knot $T(k_2, -(e+1)k_2-1)$. This completes the proof of Theorem 1.2. □

Example 3.2. Put $e = 1, k_1 = k_2 = 2$, and let x_1, x_2 be positive integers. Then by the above arguments, $p = 7x_1 + 2x_2, q = 4x_1 + x_2, r = 6x_1 + x_2$ and $T(p, q; r, -1)$ is the x_1-string fusion of $T(x_1 + x_2, 2x_1 + x_2)$ and $T(2x_1, -5x_1)$. Hence by putting $x_1 = 2, x_2 = 3$, we see that $T(20, 11; 15, -1)$ is the 2-string fusion of $T(5, 7)$ and $T(4, -10)$, and this is the example in Example 1.3. If we put $x_1 = 2$ and $x_2 = 1$, then we see that $T(16, 9; 13, -1)$ is the 2-string fusion of $T(3, 5)$ and $T(4, -10)$. This is the smallest example of our knots.

4. Essential Tangles

Let p, q be coprime integers with $1 < p < q$, and k be an integer with $0 < k < p$. Consider the torus knot $T(p, q)$ and take an arc α which intersects k strings in the parallel p strings of $T(p, q)$ as in Fig. 9(1). Let $N(\alpha)$ be a regular neighborhood of α in S^3. Put $B = \mathrm{cl}(S^3 - N(\alpha))$ and $t(p, q; k) = \mathrm{cl}(T(p, q) - N(\alpha))$. Then the pair $(B, t(p, q; k))$ is a k-string tangle as in Fig. 9(2).

Lemma 4.1. *The tangle $(B, t(p, q; k))$ has a knotted component.*

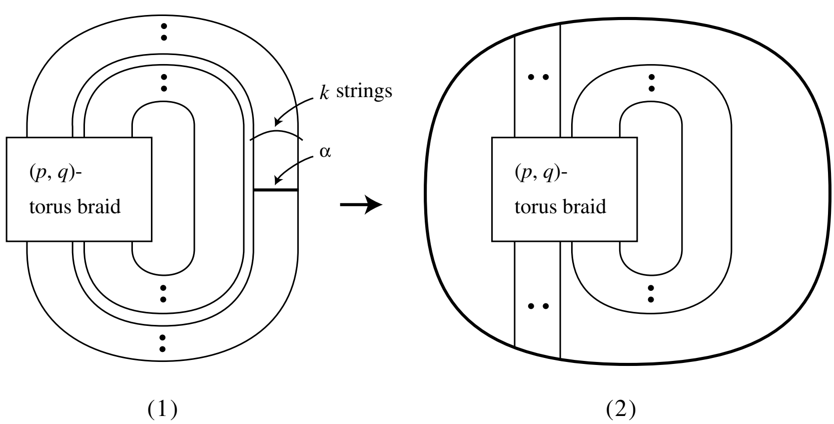

Fig. 9. Tangle $(B, t(p, q; k))$.

Proof. Put $t(p,q;k) = t_1 \cup t_2 \cup \cdots \cup t_k$. By $p < q$, we can put $q = np + m$ $(0 < m < p)$. Then, since $t_1, t_2, \ldots, t_k$ are arcs properly embedded in B each of which is a local torus knot, we can put $t_i = t(a_i, na_i + c_i; 1)$ $(i = 1, 2, \ldots, k)$, where $a_1 + a_2 + \cdots + a_k = p$, $c_1 + c_2 + \cdots + c_k = m$ and we have $q = n(a_1 + a_2 + \cdots + a_k) + (c_1 + c_2 + \cdots + c_k)$.

Suppose $a_i = 1$ for all $i = 1, 2, \ldots, k$. Then $p = a_1 + a_2 + \cdots + a_k = k < p$. This contradiction shows that there is at least one i_0 with $a_{i_0} > 1$. Then, since $n > 0$ and $c_{i_0} > 0$, the arc $t_{i_0} = t(a_{i_0}, na_{i_0} + c_{i_0}; 1)$ is a knotted component because of $na_{i_0} + c_{i_0} > a_{i_0} > 1$. □

Proposition 4.2. *The tangle* $(B, t(p,q;k))$ *is an essential tangle.*

Proof. Suppose the tangle $(B, t(p,q;k))$ is inessential. Then, by the definition of essential tangles, there is a disk properly embedded in B which separates those components. Then we may assume that the disk splits those components into two classes $t_1 \cup \cdots \cup t_j$ and $t_{j+1} \cup \cdots \cup t_k$ and that the knotted component t_{i_0} of Lemma 4.1 is contained in $t_1 \cup \cdots \cup t_j$. Consider the 2-string tangle $(B, t_{i_0} \cup t_k)$. Then, since t_{i_0} is a knotted component and it is split from t_k, $\mathrm{cl}(B - N(t_{i_0} \cup t_k))$ is not a handlebody. However, since torus knots or links have tunnel number one and the arc connecting adjacent two strings is an unknotting tunnel by [1], we see that $(B, t_{i_0} \cup t_k)$ is a free tangle and $\mathrm{cl}(B - N(t_{i_0} \cup t_k))$ is a genus two handlebody. This contradiction completes the proof. □

Proposition 4.3. *For any positive integer* $x > 0$, *the tangle* $(B, t(xp, -xq; x))$ *is an essential tangle.*

Proof. Since $t(p, -q; 1)$ is a knotted arc properly embedded in B, the tangle $(B, t(p, -q; 1))$ is an essential tangle. Then by replacing the arc with x strings, we see that $(B, t(xp, -xq; x))$ is an essential tangle. □

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