Tunnel number, 1-bridge genus and h-genus of knots

by

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Abstract. In the present paper, we introduce three geometric invariants of knots K : $t(K)$, $g_1(K)$, $h(K)$, and study the relationship among these invariants, connected sum and tangle decompositions.

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1. Introduction

Let M be an orientable closed 3-manifold. Then it is well known that M can be decomposed into two handlebodies, say V_1, V_2, so that $M = V_1 \cup V_2$ and $V_1 \cap V_2 = \partial V_1 = \partial V_2$. This is called a Heegaard splitting of M, denoted by (V_1, V_2), and $\partial V_1 (= \partial V_2)$ is called a Heegaard surface of M. Then the genus of V_1 (= the genus of V_2) is called the Heegaard genus of the Heegaard splitting (V_1, V_2), and we define the Heegaard genus of M as the minimal genus among all Heegaard splittings of M. Let K be a knot in the 3-sphere S^3. Then the following facts are well known :

Fact 1.1 *There is a Heegaard splitting (V_1, V_2) of S^3 such that a handle of V_1 contains K as a core of V_1.*

Fact 1.2 *There is a Heegaard splitting (V_1, V_2) of S^3 such that each handlebody V_i intersects K in a single trivial arc in V_i $(i = 1, 2)$. We call such a Heegaard splitting a 1-bridge decompositin of K.*

Fact 1.3 *There is a Heegaard splitting (V_1, V_2) of S^3 such that K is contained in the Heegaard surface $\partial V_1 = \partial V_2$.*

By these facts, the following geometric invariants are define :

Definition 1.4 (tunnel number) *We define $t(K)$ as the minimal genus -1 among all Heegaard splittings satisfying the condition in Fact 1.1, and we call $t(K)$ the tunnel number of K.*

Definition 1.5 (1-bridge genus) *We define $g_1(K)$ as the minimal genus among all Heegaard splittings satisfying the condition in Fact 1.2, and we call $g_1(K)$ the 1-bridge genus of K.*

Definition 1.6 (h-genus) *We define $h(K)$ as the minimal genus among all Heegaard splittings satisfying the condition in Fact 1.3, and we call $h(K)$ the h-genus of K.*

Remark 1 The original definition of the tunnel numer is the minimal number among all unknotting tunnel systems of a given knot K in S^3, where an unknotting tunnel system is a family of mutually disjoint arcs in S^3, say Γ, such that $\Gamma \cap K = \partial\Gamma$ and the exterior of $\Gamma \cup K$ is homeomorphic to a handlebody. Then, since the regular neighborhood of $\Gamma \cup K$ is homeomorphic to a handlebody, we see that Definition 1.1 is equivalent to the original definition.

In the present paper, we study the relationship among these invariants, connected sum and tangle decompositions. Throughout the present paper, we work in the piecewise linear category. For a manifold X and subcomplex Y in X, we denote a regular neighborhood of Y in X by $N(Y, X)$ or $N(Y)$ simply.

2. Basic relation

Proposition 2.1 $t(K) \leq g_1(K) \leq h(K) \leq t(K) + 1$ *for any knot K in S^3.*

Proof. Let (V_1, V_2) be the Heegaard splitting of S^3 which attains the 1-bridge genus $g_1(K)$, i.e., $V_i \cap K$ is a single trivial arc in V_i $(i = 1, 2)$ and the genus of V_1 is equal to $g_1(K)$. Let $N(V_2 \cap K)$ be a regular neighborhood of the arc $V_2 \cap K$ in V_2, and put $W_1 = V_1 \cup N(V_2 \cap K)$, $W_2 = cl(V_2 - N(V_2 \cap K))$. Then (W_1, W_2) is a Heegaard splitting of S^3 and W_1 contains K as a core of a handle of W_1. Then by Denifition 1.4, $t(K) \leq g(W_1) - 1 = g(V_1) = g_1(K)$, where $g(\ \cdot\ )$ is the genus of the handlebody.

Let (V_1, V_2) be the Heegaard splitting of S^3 which attains the h-genus of K, i.e., $\partial V_1 = \partial V_2$ contains K and $g(V_1)$ is equal to $h(K)$. Consider K as a union of two arcs k_1 and k_2 so that $K = k_1 \cup k_2$ and $k_1 \cap k_2 = \partial k_1 = \partial k_2$. Put the interior of k_i into the interior of V_i $(i = 1, 2)$. Then the Heegaard splitting (V_1, V_2) can be regarded as a 1-bridge decomposition of K, and hence $g_1(K) \leq g(V_1) = h(K)$.

Let (V_1, V_2) be the Heegaard splitting of S^3 which attains the tunnel number of K, i.e., V_1 conatins K as a core of a handle of V_1 and $g(V_1) - 1$ is equal to $t(K)$. Since K is a core of a handle of V_1, we can isotope K into the Heegaard surface $\partial V_1 = \partial V_2$, and we can regard K as a knot in the Heegaard surface. Hence $h(K) \leq g(V_1) = t(K) + 1$. This completes the proof of Proposition 2.1. □

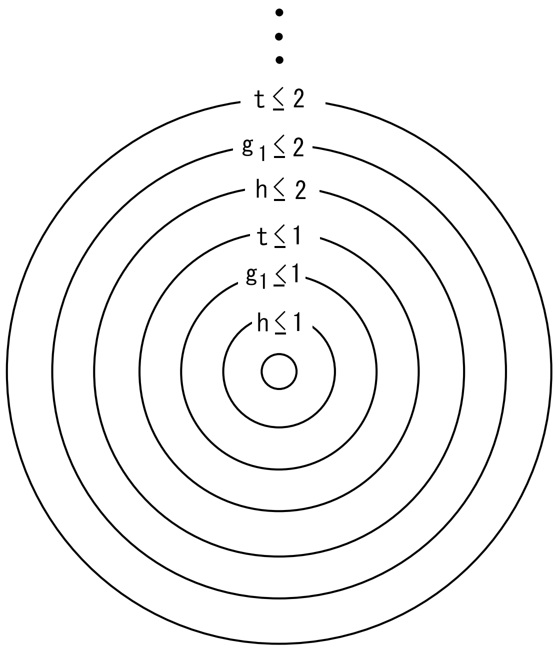

Figure 1

By Proposition 2.1 we get the Venn diagram as in Figure 1. For each positive integer n, we define the families of knots A_n, B_n and C_n between adjacent two circles in the Venn diagram as follows :

Definition 2.2 (1) *A_n is the family consisting of all knots K with $n-1 < t(K)$ and $h(K) \leq n$,*
(2) *B_n is the family consisting of all knots K with $n < h(K)$ and $g_1(K) \leq n$,*
(3) *C_n is the family consisting of all knots K with $n < g_1(K)$ and $t(K) \leq n$.*

Then it is conjectured that A_n, B_n, C_n are all non-empty families for all n. However, all we know are few cases as follows :
A_1 contains all (non-trivial) torus knots.
B_1 contains all (non-torus) 2-bridge knots.
C_1 contains all knots introduced in [Theorem 2.1 MSY1].
A_2 conatins all (p, q, r)-pretzel knots with $|p| > 2, |q| > 2, |r| > 2$ by [Theorem 2.2 MSY2].
B_2 contains all knots introduce in [M3], or (more genelally) all knots in F(1-2)

introduce in the present paper by [Theorem 1.6 M7]

The author does not know any results further than these, but by these results we can see that $t(K), g_1(K)$ and $h(K)$ are all mutually different invariants.

3. Additivity under connected sum

Let K_1 and K_2 be two knots in S^3. Consider K_i as a knot in a 3-sphere S_i^3 ($i = 1, 2$), and let B_i be a 3-ball in S_i^3 such that $B_i \cap K_i$ is a single trivial arc in B_i. Put $S^3 = cl(S_1^3 - B_1) \cup cl(S_2^3 - B_2)$ by glueing ∂B_1 and ∂B_2 together, and put $K = cl(K_1 - B_1) \cup cl(K_2 - B_2)$ by glueing $\partial(B_1 \cap K_1)$ and $\partial(B_2 \cap K_2)$ together. Then K is a knot in the 3-sphere S^3, and we call K the connected sum of K_1 and K_2 and denot it by $K_1 \# K_2$. Then we have the following :

Fact 3.1 $t(K_1 \# K_2) \leq t(K_1) + t(K_2) + 1$ *for any two knots* $K_1,\ K_2$ *in* S^3.

Proof. For $i = 1, 2$, let (V_1^i, V_2^i) be the Heegaard splitting of S_i^3 which attain the tunnel number of K_i, i.e., V_1^i contains K_i as a core of a handle and $g(V_1^i) = t(K_i) + 1$. Take a meridian disk, say D_i, of V_1^i which intersects K_i in a single point. Put $B_i = D_i \times I$ with $I = [0, 1]$ and $D_i = D_i \times \{\frac{1}{2}\}$. Then B_i intersects K_i in a single trivial arc in B_i. Put $A_1 = B_1 \cap V_2^1 = \partial B_1 \cap \partial V_2^1 = \partial D_1 \times I$, then A_1 is an annulus in ∂V_2.

To make A_1 primitive, take an arc, say α, properly embedded in V_2^1 which is properly isotopic to an essential arc in A_1, and take a regular neighborhood of α in V_2^1, say $N(\alpha)$. Put $V_1 = cl(V_1^1 - B_1) \cup cl(V_1^2 - B_2) \cup N(\alpha)$, where $D_1 \times \partial I$ is identified with $D_2 \times \partial I$ and $\partial(B_1 \cap K_1)$ is identified with $\partial(B_2 \cap K_2)$, then V_1 is a handlebody. On the otherhand put $V_2 = cl(V_2^1 - N(\alpha)) \cup V_2^2$, where A_1 is identified with the annulus $A_2 = B_2 \cap V_2^2$, then since A_1 is primitive in the handlebody $cl(V_2^1 - N(\alpha))$, V_2 is a handlebody too. Hence (V_1, V_2) is a Heegaard splitting of S^3, V_1 conatins $K_1 \# K_2$ as a core of a handle of V_1 and $g(V_1) = g(V_1^1) + g(V_1^2)$. Thus we have $t(K_1 \# K_2) \leq g(V_1) - 1 = g(V_1^1) + g(V_1^2) - 1 = (t(K_1) + 1) + (t(K_2) + 1) - 1 = t(K_1) + t(K_2) + 1$. □

Fact 3.2 $g_1(K_1 \# K_2) \leq g_1(K_1) + g_1(K_2)$ *for any two knots* $K_1,\ K_2$ *in* S^3.

Proof. For $i = 1, 2$, let (V_1^i, V_2^i) be the Heegaard splitting of S_i^3 whidh attains the 1-bridge genus of K_i, and let B_i be a 3-ball in S_i^3 such that $B_i \cap K_i$ is a single trivial arc in B_i and both $B_i \cap V_1^i$ and $B_i \cap V_2^i$ are hemiballs. Put $V_1 = cl(V_1^1 - B_1) \cup cl(V_1^2 - B_2)$ and $V_2 = cl(V_2^1 - B_1) \cup cl(V_2^2 - B_2)$, where $\partial B_1 \cap V_j^1$ is identified with $\partial B_2 \cap V_j^2$ ($j = 1, 2$) and $\partial(B_1 \cap K_1)$ is idetified with $\partial(B_2 \cap K_2)$. Then (V_1, V_2) is Heegaard splitting of S^3 which gives a 1-bridge decomposition of $K_1 \# K_2$. Hence $g_1(K_1 \# K_2) \leq g(V_1) = g(V_1^1) + g(V_1^2) = g(K_1) + g(K_2)$. □

Fact 3.3 $h(K_1 \# K_2) \leq h(K_1) + h(K_2)$ *for any two knots* $K_1,\ K_2$ *in* S^3.

Proof. This is proved by the argument similar to the proof of Fact 3.2. □

Concerning the estimate of these invariants under connected sum, we haver already shown in [Ko] and [M3] that for any integer $n > 0$ there are infinitely many pairs of knots K_1, K_2 with $t(K_1 \# K_2) < t(K_1) + t(K_2) - n$, $g_1(K_1 \# K_2) < g_1(K_1) + g_1(K_2) - n$ and $h(K_1 \# K_2) < h(K_1) + h(K_2) - n$. However, we can ask what is the condition for these invariants not to go down under connencted sum. To answer this question, we introduce the notion of smallness and meridional smallness of knots.

Definition 3.4 (small knot) *We say that a knot K is small if $E(K)$ contains no closed essential surfaces, where $E(K) = cl(S^3 - N(K))$ is the exterior of K, $N(K)$ is the regular neighborhood of K and " essential " means " incompressible and non-peripheral ".*

Definition 3.5 (meridionally small knot) *We say that a knot K is meridionally small if $E(K)$ contains no meridional essential surfaces, where a surface $F \subset E(K)$ is meridional if $\partial F \neq \emptyset$ and each component of ∂F is a meridian loop of K.*

Suppose K is small. If K is not meridionally small, then $E(K)$ contains a meridional essential surface, and the meridian loop in $\partial E(K)$ is a boundary slope in the sense of [CGKS]. Then by [Theorem 2.0.3 CGLS], we have (i) S^3 is a Haken manifold, (ii) S^3 is a connected sum of two lens spaces, (iii) $E(K)$ contains a closed essential surface or (iv) $E(K)$ is a planar surface bundle over a circle so that a boundary slope of the fiber is a meridian loop. However, (i), (ii), (iv) do not occur because the ambient space is S^3 and (iii) contradicts the hypothesis. Hence K is meridionally small. However, the next proposition shows that there is a big difference between smallness and meridional smallness.

Proposition 3.6 ([Proposition 1.6 M5]) *For any integer $n > 0$, there are infinitely many knots K such that :*

(1) *K is meridionally small,*

(2) *K is not small,*

(3) *$t(K) > n$, $g_1(K) > n$ and $h(K) > n$.*

Concerning the condition for our invariants not to go down, we showed :

Theorem 3.7 ([Theorem 1.1 M5]) *If both K_1 and K_2 are meridionally small, then $t(K_1 \# K_2) \geq t(K_1) + t(K_2)$.*

The estimate in Theorem 3.7 is best possible, because if K_1 and K_2 are 2-bridge knots then $t(K_1 \# K_2) = t(K_1) + t(K_2)$ and both K_1 and K_2 are meridionally small by [Theorem HT]. Next on the 1-bridge genus, P. Hoidn showed :

Theorem 3.8 ([Theorem Ho]) *If both K_1 and K_2 are small, then $g_1(K_1 \# K_2) \geq g_1(K_1) + g_1(K_2) - 1$.*

To this theorem, it seems natural to ask if it is possible to change the condition " small " to the condition " meridionally small ". To answer this question, we need the following :

Theorem 3.9 ([Theorem 1.6 M6]) *Suppose both K_1 and K_2 are meridionally small. Then $t(K_1 \# K_2) = t(K_1) + t(K_2) + 1$ if and only if $g_1(K_i) = t(K_i) + 1$ for both $(i = 1, 2)$.*

Then we get :

Theorem 3.10 *If both K_1 and K_2 are meridionally small, then $g_1(K_1 \# K_2) \geq g_1(K_1) + g_1(K_2) - 1$.*

Proof. We devide the proof into two subcases.

Case 1 : $g_1(K_i) = t(K_i) + 1$ for both $i = 1, 2$. Then, by Proposition 2.1 and Theorem 3.9, we have : $g_1(K_1 \# K_2) \geq t(K_1 \# K_2) = t(K_1) + t(K_2) + 1 = (g_1(K_1) - 1) + (g_1(K_2) - 1) + 1 = g_1(K_1) + g_1(K_2) - 1$.

Case 2 : For at least one of K_1 and K_2, say K_1, $g_1(K_1) = t(K_1)$ holds. Then, by Proposition 2.1 and Theorem 3.7, we have : $g_1(K_1 \# K_2) \geq t(K_1 \# K_2) \geq t(K_1) + t(K_2) \geq g_1(K_1) + g_1(K_2) - 1$. This completes the proof of Theorem 3.10. □

On the best possibility of the estimate in Theorem 3.10, the author does not know anything at all. Hence we ask :

Question 3.11 *Is the estimate in Theorem* 3.10 *best possible* ?

Remark 2 The author believes that the estimate is best possible. But to show the best possibility we see, by the proof of Theorem 3.10, that we need to find meridionally small knots K with $g_1(K) = t(K) + 1$.

Concerning the h-genus of knots we have :

Proposition 3.12 (1) *If both K_1 and K_2 are meridionally small, then $h(K_1 \# K_2) \geq h(K_1) + h(K_2) - 2$.*

(2) *There are infinitely many pairs of meridionally small knots K_1 and K_2 with $h(K_1 \# K_2) = h(K_1) + h(K_2) - 1$.*

Proof. (1) By Proposition 2.1 and Theorem 3.7, we have : $h(K_1 \# K_2) \geq t(K_1 \# K_2) \geq t(K_1) + t(K_2) \geq (h(K_1) - 1) + (h(K_2) - 1) \geq h(K_1) + h(K_2) - 2$. (2) Let K_1 and K_2 be non-torus 2-bridge knots. Then $h(K_1) = h(K_2) = 2$, and by [Theorem M2] we have $h(K_1 \# K_2) = 3$. Hence $h(K_1 \# K_2) = h(K_1) + h(K_2) - 1$ and both K_1 and K_2 are meridionally small by [Theorem HT]. □

Hence we can ask :

Question 3.13 *Is the estimate in Proposition* 3.12(1) *best possible* ?

4. Characterization of composite knots

We say that a knot K in the 3-sphere S^3 is composite if K is a connected sum of non-trivial two knots, and that K is prime if K is not composite. Then on the primeness, the first resut is :

Proposition 4.1 ([MS, No, Sc]) *Tunnel number one knots are prime.*

This implies that 1-bridge genus one knots are prime and that h-genus one knots are prime. Contrary to Proposition 4.1, there are infinitely many composite knots with tunnel number 2, 1-bridge genus 2 and h-genus 2. So we can ask what kind of composite knots have those invariants 2. The first answer is :

Therem 4.2 ([Theorem M2]) *Let K_1 and K_2 be non-trivial knots in S^3. Suppose $h(K_1 \# K_2) = 2$, then $h(K_1) = h(K_2) = 1$, i.e., both K_1 and K_2 are torus knots.*

To consider the cases of the other invariants, we need some notations and terms concerning tangles. Let B be a 3-ball and $t_1 \cup t_2$ two arcs properly embedded in B. Then we say that $(B, t_1 \cup t_2)$ is a 2-string tangle.

Definition 4.3 (1) *$(B, t_1 \cup t_2)$ is essential if $cl(\partial B - N(t_1 \cup t_2))$ is incompressible in $cl(B - N(t_1 \cup t_2))$, where $N(t_1 \cup t_2)$ is a regular neighborhood of $t_1 \cup t_2$ in B.*
(2) *$(B, t_1 \cup t_2)$ is free if $cl(B - N(t_1 \cup t_2))$ is a genus two handlebody.*
(3) *$(B, t_1 \cup t_2)$ has unknotted component if at least one of (B, t_1) or (B, t_2) is a trivial ball pair.*

Using these terms and notations, we define a family of knots as follows :

Definition 4.4 *Let $F(1)$ be a family of all knots K with 2-string essential free tangle decompositions $(S^3, K) = (B_1, t_1^1 \cup t_2^1) \cup (B_2, t_1^2 \cup t_2^2)$ such that at least one of $(B_1, t_1^1 \cup t_2^1)$ and $(B_2, t_1^2 \cup t_2^2)$ has an unknotted component.*

Figure 2 is an example of a 2-string essential free tange with one unknotted component. Then concerning composite knots with tunnel number 2, we showed :

Theorem 4.5 ([Theorem M1, Theorem 0.4 M4]) *Let K_1 and K_2 be non-*

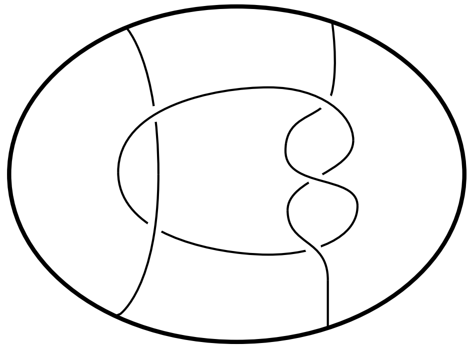

Figure 2

trivial knots in S^3. Suppose $t(K_1 \# K_2) = 2$, then one of the follwoing holds :
(1) *$t(K_1) = t(K_2) = 1$ and $g_1(K_i) = 1$ for at least one of $i = 1, 2$,*
(2) *One of K_1 and K_2, say K_1, is a 2-bridge knot and K_2 belongs to $F(1)$.*

Remark 3 By [Theorem 0.4 M4], we see that $t(K) = 2$ if K belongs to $F(1)$.

Next consider the subfamilies F(1-1) and F(1-2) of $F(1)$ as follows :

Definition 4.6 (1) *Let F(1-1) be a family of all knots K with 2-string essential free tangle decompositions $(S^3, K) = (B_1, t_1^1 \cup t_2^1) \cup (B_2, t_1^2 \cup t_2^2)$ such that one of $(B_1, t_1^1 \cup t_2^1)$ and $(B_2, t_1^2 \cup t_2^2)$ has unknotted component and the other has no.*
(2) *Let F(1-2) be a family of all knots K with 2-string essential free tangle decompositions $(S^3, K) = (B_1, t_1^1 \cup t_2^1) \cup (B_2, t_1^2 \cup t_2^2)$ such that both $(B_1, t_1^1 \cup t_2^1)$ and $(B_2, t_1^2 \cup t_2^2)$ have unknotted components.*

By these definitions, we have $F(1) = F(1\text{-}1) \cup F(1\text{-}2)$ and $F(1\text{-}1) \cap F(1\text{-}2) = \emptyset$. Then we showed :

Theorem 4.7 ([Theorem 1.6 M7]) *Let K_1 and K_2 be non-trivial knots in S^3. Suppose $g_1(K_1 \# K_2) = 2$, then one of the following holds :*
(1) *$g_1(K_1) = g_1(K_2) = 1$,*
(2) *One of K_1 and K_2, say K_1, is a 2-bridge knot, $t(K_2) = 1$ and $g_1(K_1) = 2$,*
(3) *One of K_1 and K_2, say K_1, is a 2-bridge knot and K_2 belongs to F(1-2).*

Remark 4 By [Theorem 1.6 M7], we see that $t(K) = 2$ and $g_1(K) = 2$ if K belongs to F(1-2).

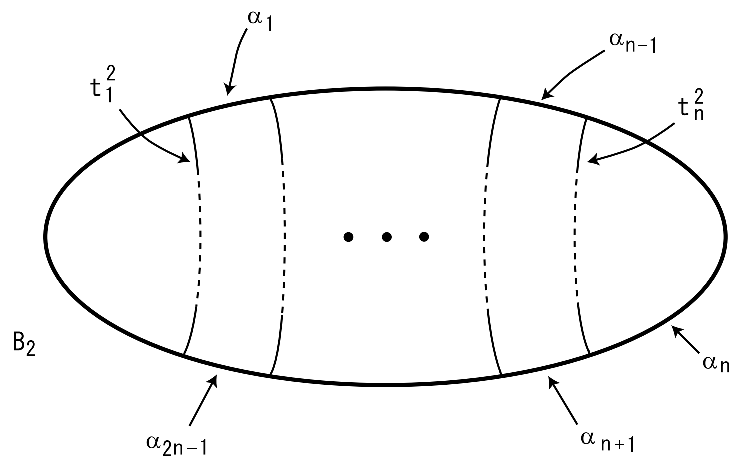

Figure 3

Finally consider the following family :

Definition 4.8 *Let $F(0)$ be a family of all knots K with 2-string essential free tangle decompositions $(S^3, K) = (B_1, t_1^1 \cup t_2^1) \cup (B_2, t_1^2 \cup t_2^2)$ such that neither $(B_1, t_1^1 \cup t_2^1)$ nor $(B_2, t_1^2 \cup t_2^2)$ has unknotted component.*

In general, consider knots K in S^3 with n-string free tangle decompositions, i.e., $(S^3, K) = (B_1, t_1^1 \cup t_2^1 \cup \cdots \cup t_n^1) \cup (B_2, t_1^2 \cup t_2^2 \cup \cdots \cup t_n^2)$ and $cl(B_i - N(t_1^i \cup t_2^i \cup \cdots \cup t_n^i))$ is a genus n handlebody for both $i = 1, 2$. Then we have :

Proposition 4.9 *If K has an n-string free tangle decomposition, then $t(K) \leq 2n - 1$.*

Proof. Let $\alpha_1, \alpha_2, \cdots, \alpha_{2n-1}$ be the arcs in ∂B_2 each of which connects two points of $\partial(t_1^2 \cup t_2^2 \cup \cdots \cup t_n^2)$ as indicated in Figure 3.

Let $N(K)$ be a regular neighborhood of K in S^3 and $N(\alpha_i)$ a regular neighborhood of α_i in B_2. Put $W_1 = N(K) \cup N(\alpha_1) \cup \cdots \cup N(\alpha_n)$. Then W_1 is a genus $2n$ handlebody.

Put $W_2 = cl(S^3 - W_1)$. Then $W_2 = cl(B_1 - N(K)) \cup cl(B_2 - N(K) - (N(\alpha_1) \cup \cdots \cup N(\alpha_n)))$ and the intersection in this union is a 2-disk because $\alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_{2n-1}$ is a single arc in ∂B_2. Then, since both $cl(B_1 - N(K))$ and $cl(B_2 - N(K) - (N(\alpha_1) \cup \cdots \cup N(\alpha_n)))$ are genus n handlebodies, W_2 is a genus $2n$ handlebody. This shows that (W_1, W_2) is a genus $2n$ Heegaard splitting of S^3 and W_1 contains K as a core of a handle. This shows that $t(K) \leq g_1(W_1) - 1 = 2n - 1$. □

Together with Proposition 2.1, Remark 3, Remark 4 and this proposition, we have:

$K \in F(0) \Longrightarrow 2 \le t(K) \le 3$ and $2 \le g_1(K) \le 4$
$K \in F(\text{1-1}) \Longrightarrow t(K) = 2$ and $2 \le g_1(K) \le 3$
$K \in F(\text{1-2}) \Longrightarrow t(K) = 2$ and $g_1(K) = 2$

Hence we close the present paper with the following problem and question :

Problem 4.10 (1) *Determine the tunnel number and the* 1*-bridge genus of knots in* $F(0)$.
(2) *Determine the* 1*-bridge genus of knots in* $F(\text{1-1})$.

Question 4.11 *Is the estimate in Proposition* 4.9 *best possible* ?

Remark 5 The author has no example K with $K \in F(0)$ and $t(K) = 3$, but conjectures that $t(K) = 3$ for any knot $K \in F(0)$. This implies the best possibility of the estimate in Proposition 4.9 for the case when $n = 2$.

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