A note on essential tori in the exteriors of torus knots with twists

by

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Abstract. In the present note, we will show that there are infinitely many torus knots with twists containing essential tori in the exteriors.

Key words and phrases : torus knots, twists, essential tori
2000 Mathematics Subject Classification : 57M25, 57N10

1. Introduction

In the present note, we study essential tori in the exteriors of torus knots with twists. In particular, we will show that there are infinitely many torus knots with full twists on r-strands containing essential tori in the exteriors for any composite number r, and that those knots are cable knots along some torus knots. We note that those knots treated in the present note have been quoted as twisted torus knots in [CDW].

Some of our examples are related to "unexpected" Dehn surgery in classical sense. In fact, any torus knots and some 2-cable knots of torus knots yield lens spaces (by certain surgery coefficients), which was found by Moser in [Ms] and by Bailey–Rolfsen in [BR] respectively. Their research has grown up as *lens space surgery* (and exceptional Dehn surgery), and main target is now hyperbolic knots, after the first hyperbolic example (ex. the famous pretzel knot $P(-2,3,7)$ found by Fintushel–Stern in [FS]). We note that some such hyperbolic knots, including $P(-2,3,7)$ (as $K(5,3;2,1)$), are constructed as in Theorem 1.1. We are concerned with the "boundary" between hyperbolic knots and satellite knots in the classification of knots.

On the other hand, those knots treated in the present note are related to the additivity problem of tunnel number or 1-bridge genus of knots under connected sum. In fact, those knots obtained from torus knots by adding full twists on 2-strands have been used to show the supper additivity of tunnenl number in [MSY], and such knots have also been used to show the best possibility of Hoiden-Morimoto's inequality concerning 1-bridge genus for meridionally small knots in [Ho, M1, M2].

Therefore we are interested in those knots treated in the present note and we consider that those knots are important objects in study of Dehn surgery and additivity problem of tunnel number.

Let p, q, r, s be integers such that $\gcd(p, q) = 1$, $p > r > 1$ and $q > 0$, and let $T(p, q)$ be the torus knot of type (p, q) in S^3, for $T(p, q)$ we refer [Ro]. Then, add s-times full twists on mutually parallel r-strands to $T(p, q)$, and denote the knot obtained by this operation by $K(p, q; r, s)$ as illustrated in Figure 1. It is well known that $T(p, q)$ contains no closed essential surfaces in the exterior. Hence as a generalization of this fact, we study if $K(p, q; r, s)$ contains closed essential surfaces in the exterior or not. Concerning this problem, the first author announced the following :

Theorem 1.1 ([M2]) Suppose $r = 2$. Then $K(p, q; 2, s)$ contains no closed essential surfaces or no meridionally essential surfaces in the exterior for any s and coprime integers p, q. Note that $K(p, q; 2, s)$ has been denoted by $K(p, q; r)$ in [MSY, M2].

K(p, q ; r, s)

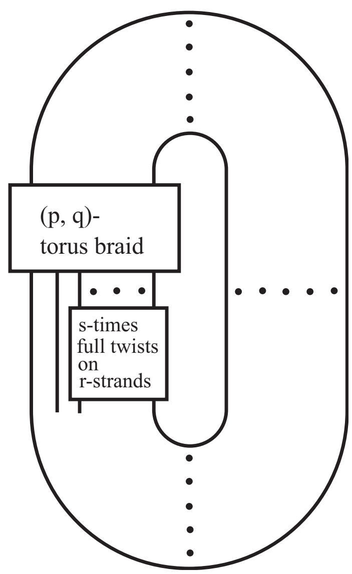


Figure 1

In the present note, we show that for any composite number r, there are infinitely many $K(p, q; r, s)$ containing essential tori in the exteriors (Theorem 2.2). Hence we can ask the following :

Question

(1) Do there exist $K(p,q;r,s)$ containing closed essential surfaces in the exteriors for prime number $r > 2$?

(2) Do there exist $K(p,q;r,s)$ containing closed essential surfaces (not tori) in the exteriors for composite number r ?

Throughout the present note, we will work in the piecewise linear category. For the definitions of standard terms in 3-dimensional topology and knot theory, we refer [He] or [Ro].

2. Theorem and Proof

Let k be an integer with $k > 1$, and consider two k-string braids $(\sigma_1\sigma_2\cdots\sigma_{k-1})^k$ and $(\sigma_{k-1}\sigma_{k-2}\cdots\sigma_1)^k$, where σ_i is the permutation of the i-th string and the $(i+1)$-th string as in Figure 2. Then we have :

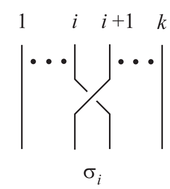


Figure 2

Lemma 2.1 For any $k > 1$, the two braids $(\sigma_1\sigma_2\cdots\sigma_{k-1})^k$ and $(\sigma_{k-1}\sigma_{k-2}\cdots\sigma_1)^k$ are the same ones as elements in the k-string braid group.

Proof. First take the braid $(\sigma_1\sigma_2\cdots\sigma_{k-1})^k$, and deform the $(k-1)$-th string as in Figure 3, which is the case when $k = 5$. Next deform the $(k-2)$-th string as in Figure 3, and continue these deformations up to the 2nd-string. Then we get the braid $(\sigma_{k-1}\sigma_{k-2}\cdots\sigma_1)^k$. This deformation can be realized by the braid relations : $\sigma_i\sigma_j = \sigma_j\sigma_i$ ($|i-j| > 2$) and $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ ($i = 1, 2, \cdots, k-1$). Hence we have $(\sigma_1\sigma_2\cdots\sigma_{k-1})^k = (\sigma_{k-1}\sigma_{k-2}\cdots\sigma_1)^k$ as elements in the k-string braid group. □

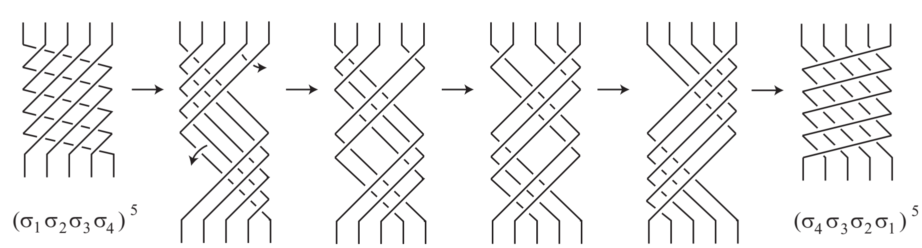
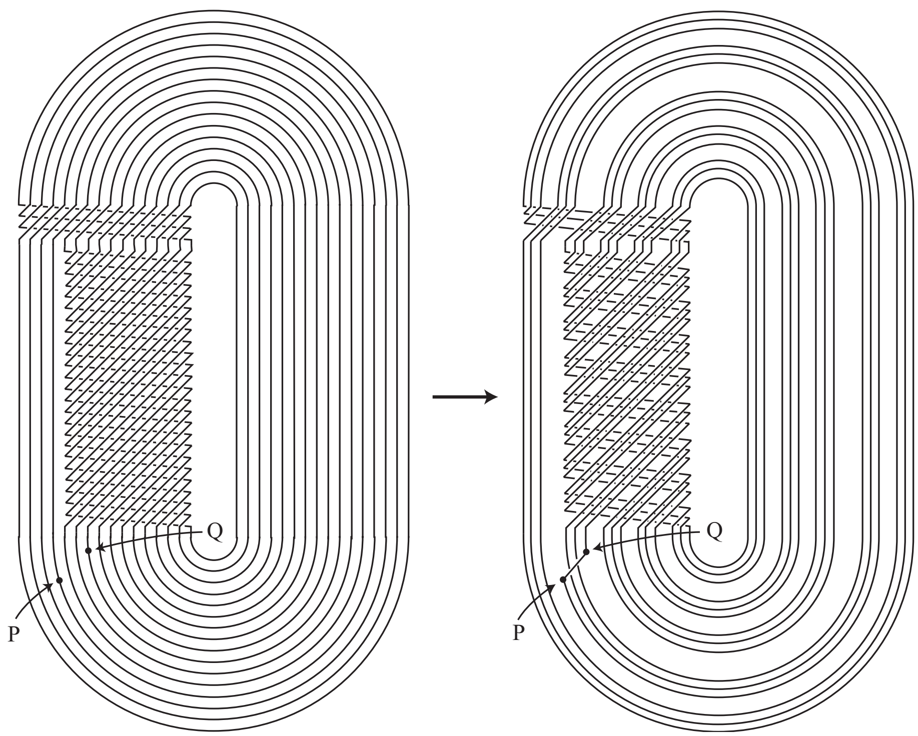


Figure 3

Let $k > 1,\ m > 1$ be integers, and put $r = km$. Then r is a composite number. Let n be an integer with $n \geq m$, and let s be an integer with $|ms+1| > 1$ (i.e., $s \neq 0$ or $ms \neq -2$ because of $m > 1$). Then by putting $p = kn+1$ and $q = k$, we have :

Theorem 2.2 $K(kn+1, k; km, s)$ is a satellite knot whose companion is the torus knot $T(m, ms+1)$, and the pattern is the torus knot $T(k, k(n+m^2s)+1)$. Hence $K(kn+1, k; km, s)$ is a k-cable knot along $T(m, ms+1)$.

Proof. Consider the knot $K(kn+1, k; km, s)$ as illustrated in Figure 4(1), which is the case when $k = 3$, $m = 4$, $n = 5$ and $s = 2$, i.e. $p = kn+1 = 16$, $q = k = 3$, $r = km = 12$ and $s = 2$. Take a point P in the $\{k(n-m)+1\}$-th string, and take a point Q in the $\{k(n-m)+1+k\}$-th string bellow the full twists. Then, as in Figure 4(2), the arc with the end points P and Q can be deformed into the bridge PQ. Regard k-strands as a single bunch, and take a solid torus V containing the bunch as in Figure 5. Let ℓ be the core of V and consider ℓ as a knot in S^3. Then ℓ is formed by s-times full twists on m-strands and a $\frac{2\pi}{m}$-twist. This means that ℓ is the torus knot $T(m, ms+1)$. Hence we see that $K(kn+1, k; km, s)$ is a satellite knot with the companion $T(m, ms+1)$, and is a k-cable knot along $T(m, ms+1)$.



(1) $K(16, 3; 12, 2)$ (2)

Figure 4

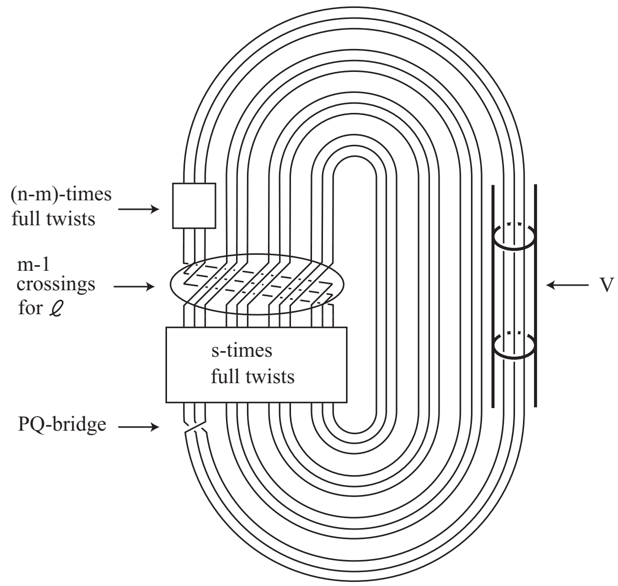


Figure 5

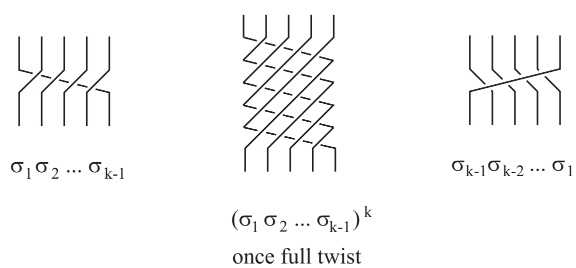


Figure 6

Next, let's detect the pattern. First we see that this pattern is formed by full twists on k-strands and the PQ-bridge. By Figure 4(2) and Figure 5, we can see that there are $(n-m+1+ms)$-times full twists in the projection, because there are $(n-m)$-times full twists in the rectangle of Figure 5, once full twists under the $(m-1)$-crossings for ℓ and ms-times full twists on k-strands in the s-times full twists on m-strands. In addition, by considering the prefered longitude of V, each crossing point of the projection of ℓ corresponds to once full twist on k-strands. Hence we need to add $\{m-1+m(m-1)s\}$-times full twists to the above full twists, because there are $(m-1)$-crossings for ℓ and $m(m-1)s$-crossings in the s times full twists on m-strands. After all we see that the pattern is formed by $(n-m+1+ms+m-1+m^2s-ms)=(n+m^2s)$-times full twists on k-strands and the PQ-bridge. By the way, once full twist corresponds to $(\sigma_1\sigma_2\cdots\sigma_{k-1})^k$ and the PQ-bridge corresponds to $(\sigma_{k-1}\sigma_{k-2}\cdots\sigma_1)$ as in Figure 6. Thus the pattern is represented by the braid $(\sigma_1\sigma_2\cdots\sigma_{k-1})^{k(n+m^2s)}(\sigma_{k-1}\sigma_{k-2}\cdots\sigma_1)$. Then, by Lemma 2.1, this braid coincides with $(\sigma_{k-1}\sigma_{k-2}\cdots\sigma_1)^{k(n+m^2s)+1}$, and the knot represented by this braid is the torus knot $T(k, k(n+m^2s)+1)$. This completes the proof of Theorem 2.2. $\square$

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