Tunnel Numbers of Knots

by

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Abstract

Tunnel number of a knot is a geometric invariant of a knot in the 3-sphere, and has interesting properties related to the connected sum of knots. In this article, we show the existence of infinitely many pairs of knots each tunnel number of which goes up under tne connected sum, and show the existence of infinitely many pairs of knots each tunnel number of which goes down under the connected sum. In addition, we study the degeneration ratio of tunnel numbers under the connected sum.

1. Introduction

In the present article, we introduce the tunnel numbers of knots in the 3-sphere S^3, and calculate the tunnel numbers of several examples. Further, we study the relation between the tunnel numbers and the connected sum of knots.

This geometric invariant, the tunnel number, is closely related to the Heegaard genus of the 3-manifolds as the knot exteriors. Therefore, in studying tunnel numbers of knots, we need not only knot theoretical technique but also 3-manifold topology argument. For knot theory we refer to Rolfsen's book [Rs] and for 3-manifold topology we refer to Hempel's book [He].

2. Definitions and examples

To define the tunnel number, we need :

Fact 2.1 *For any knot K in the 3-sphere S^3, there is an arc system $\{\gamma_1, \gamma_2, \cdots, \gamma_t\}$ in S^3 with $(\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_t) \cap K = \partial(\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_t)$ such that the exterior of the union of K and the arcs is homeomorphic to a genus $t+1$ handlebody, i.e., $cl(S^3 - N(K \cup \gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_t) \cong$ a genus $t+1$ handlebody, where $N(\cdot)$ denotes a regular neighborhood.*

Definition2.2 We call the arc system in Fact 2.1 *an unknotting tunnel system* of K.

In particular, if the system consists of a single arc, then we call the arc *an unknotting tunnel* of K.

Definition2.3 We define *the tunnel number* of K, denoted by $t(K)$, as the minimal number of the arcs among all unknotting tunnel systems of K.

Proof of Fact 2.1. Consider the projection of K. Then we can take a small arc γ_i $(i = 1, 2, \cdots, c)$ at each crossing point, where c is the crossing number of the projection.

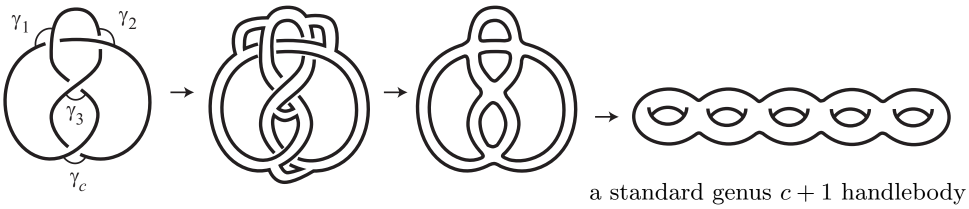

Figure 1: an unknotting tunnel system

Then, by the deformation in Figure 1, $N(K \cup \gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_c)$ is isotopic to a standard genus $c + 1$ handlebody in S^3. This means that the exterior of $N(K \cup \gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_c)$ is also a genus $c + 1$ handlebody, and completes the proof. □

Examples

(1) The trivial knot has tunnel number 0.
(2) The trefoil knot has tunnel number 1.
(3) The knot 8_{16} in Rolfsen's table ([Rs]) has tunnel number 2.
(4) Every 2-bridge knot has tunnel number 1.
(5) Every torus knot has tunnel number 1.
(6) Let p_i be an odd integer with $|p_i| > 1$ $(i = 1, 2, 3)$, and let K be the pretzel knot of type (p_1, p_2, p_3). Then K has tunnel number 2.

Proof. (1) The exterior of the trivial knot is homeomorphic to the solid torus, i.e., a genus 1 handlebody. Thus the trivial knot has tunnel number 0.
(2) By the deformation in Figure 2, we see that the arc γ is an unknottin tunnel of the trefoil knot. Thus the trefoil knot has tunnel number 1 because the trefoil knot is a non-trivial knot.

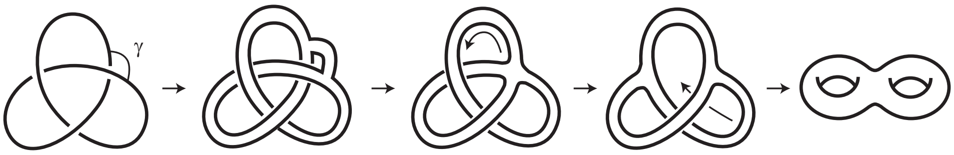

Figure 2: an unknotting tunnel

(3) The knot illustrated in Figure 3 is 8_{16} in Rolfsen's table ([Rs]). Then, by a little deformation, we see that the graph $K \cup \gamma_1 \cup \gamma_2$ becomes a trivial graph as in Figure 3. This means that the arc system $\{\gamma_1, \gamma_2\}$ is an unknotting tunnel system of 8_{16}, and hence $t(8_{16}) \leq 2$, On the other hand, by the deformation in Figure 4, we see that 8_{16} has a 2-string essential tangle decomposition, where the definition of tangle decomposition will be given after Theore 3.5. Thus, by [Sm], $t(8_{16}) \geq 2$. Thus $t(8_{16})$ has tunnel number 2.

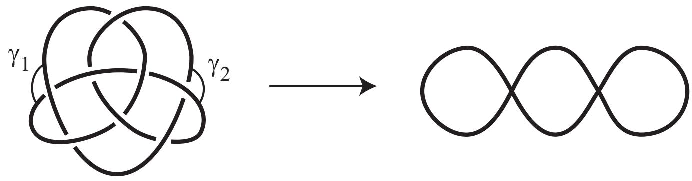

Figure 3: an unknotting tunnel system of 8_{16}

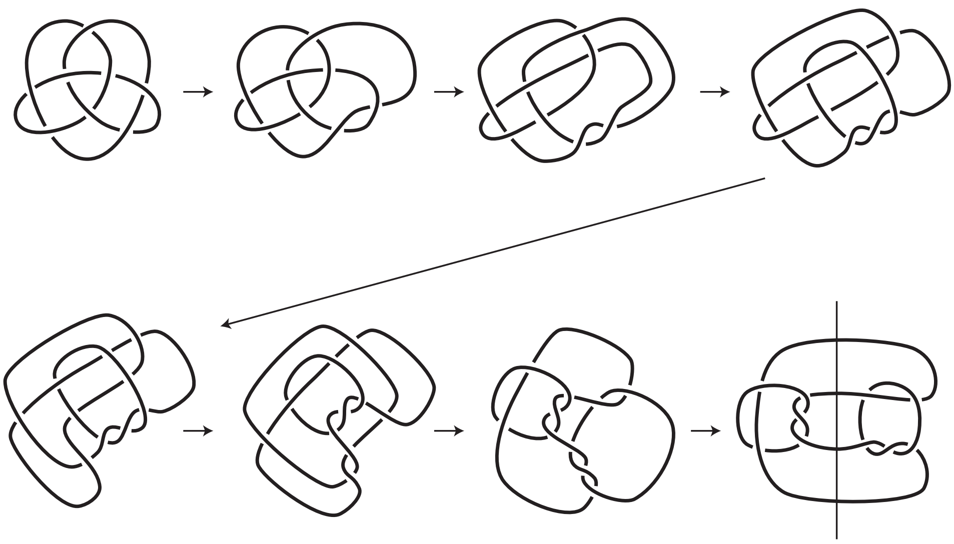

Figure 4: a 2-string essential tangle decomposition of 8_{16}

(4) Every 2-bridge knot K has a 2-string trivial tangle decomposition as in Figure 5. Then, the arc γ is an unknotting tunnel, and hence $t(K) = 1$.

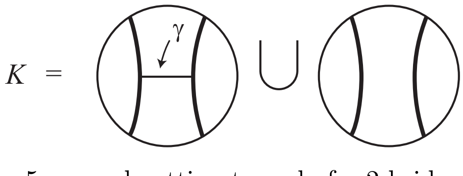

Figure 5: an unknotting tunnel of a 2-bridge knot

(5) For every torus knot K, take an arc γ indicated in Figure 6, i.e., the arc connecting

adjascent parallel two strings, where the knot illustrated in Figure 6 is the torus knot of type (5, 4). Then, by [BRZ], the arc γ is an unknotting tunnel, and hence $t(K) = 1$.

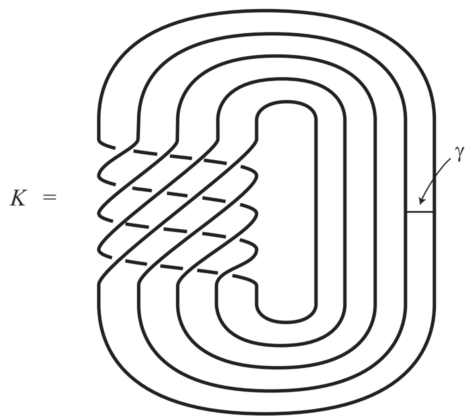

Figure 6: an unknotting tunnel of a torus knot

(6) For a pretzel knot K of type (p_1, p_2, p_3), the arc system $\{\gamma_1, \gamma_2\}$ as in Figure 7 is an unknotting tunnel system of K, where the pretzel knot illustrated in Figure 7 is of type (3, 5, 7). Then by [MSY2], we have $t(K) = 2$.

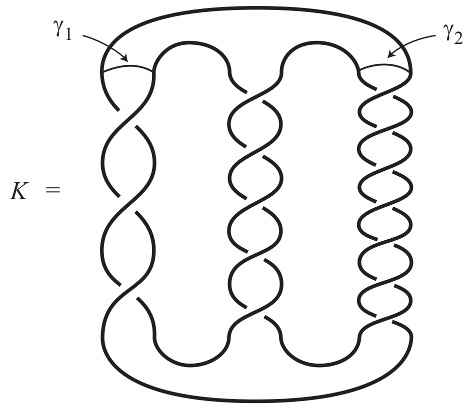

Figure 7: an unknotting tunnel system of a pretzel knot

3. Connected sum

Let K_1 and K_2 be two knots in S^3. Then the connected sum of K_1 and K_2, denoted by $K_1 \# K_2$, is defined as in Figure 8.

The behavior of geometric invariants under connected sum is a very interesting problem in knot theory. For example, for the Seifert genus $g(K)$, we have in [Rs]:

$$g(K_1 \# K_2) = g(K_1) + g(K_2)$$

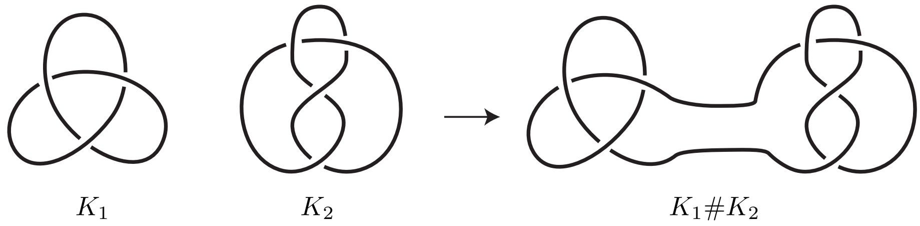

Figure 8: the connented sum

This means that the Seifert genus is additive under connected sum. For the bridge index $b(K)$, by [Sb], we have:

$$b(K_1\#K_2) = b(K_1) + b(K_2) - 1$$

In this case, $b(K) - 1$ is additive under connected sum. On the other hand, for the unknotting number $u(K)$ or th crossing number $c(K)$, it is still unknown if those invariants are additive or not.

Now, for the tunnel numbers, the most basic fact is:

Fact 3.1 $t(K_1\#K_2) \leq t(K_1) + t(K_2) + 1$

Proof. Let $\{\gamma_1, \gamma_2, \cdots, \gamma_t\}$ be the unknotting tunnel system of K_1 and $\{\delta_1, \delta_2, \cdots, \delta_s\}$ be the unknotting tunnel system of K_2. Consider the arc system $\{\gamma_1, \gamma_2, \cdots, \gamma_t, \delta_1, \delta_2, \cdots, \delta_s, \rho\}$, obtained from the union of the two unknotting systems by adding an extra arc ρ indicated in Figure 9. Then we can see that it is an unknotting tunnel system of $K_1\#K_2$. It should be noted that, in general, we need the extra arc ρ to obtain an unknotting tunnel system of the connected sum. □

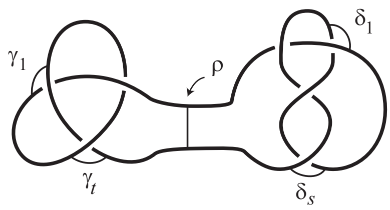

Figure 9: an unknotting tunnel system of the connented sum

In the early years of research of tunnel numbers, there were very few families of knots whose tunnel numbers were identified. In fact, we had only 2-bridge knots and torus knots. In addition, for any 2-bridge knots or any torus knots K_1, K_2, the additivity $t(K_1\#K_2) = t(K_1) + t(K_2)$ holds. Therefore, those days, the following two questions had puzzled knot theorists.

Q1 : Are there knots K_1, K_2 such that $t(K_1\#K_2) = t(K_1) + t(K_2) + 1$?

Q2 : Are there knots K_1, K_2 such that $t(K_1\#K_2) < t(K_1) + t(K_2)$?

The first result on the additivity problem of tunnel numbers is :

Theorem 3.2 ([Nw], [Sm]) *Tunnel number one knots are prime.*

This theorem says that there are no knots K_1, K_2 as in Q2 with $t(K_1\#K_2) = 1$. However, in 1990's, such knots as in Q1 and Q2 were found as follows :

Theorem 3.3 ([MSY1]) *Let K_m be the knot as in Figure 10. Then $t(K_m) = t(K_{m'}) = 1$ and $t(K_m\#K_{m'}) = 3$ for any integers m and m', i.e., "$1 + 1 = 3$".*

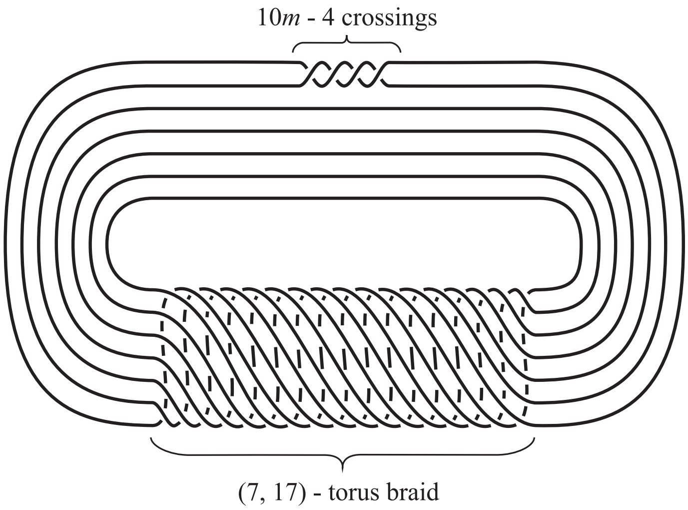

Figure 10: the knot K_m

Theorem 3.4 ([M2]) *Let K_n be the knot as in Figure 11 for any integer $n \neq 0, -1$. Then $t(K_n) = 2$ and $t(K_n\#K') = 2$ for any 2-bridge knot K', i.e., "$2 + 1 = 2$".*

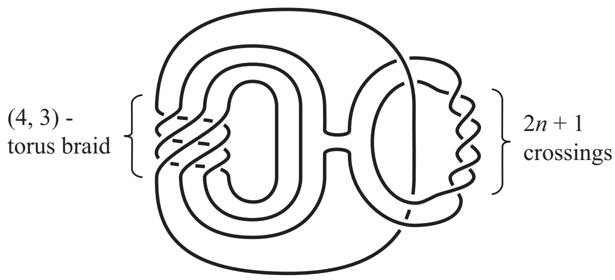

Figure 11: the knot K_n

Soon after that, we succeeded to characterize those knot types with "$2 + 1 = 2$" as follows :

Theorem 3.5 ([M3]) *Let K be a tunnel number two knot. Then, the tunnel number of $K\#K'$ is two again for any 2-bridge knot K' if and only if K has a 2-string* *essential free tangle decomposition* $(S^3, K) = (B_1, t_1^1 \cup t_1^2) \cup (B_2, t_2^1 \cup t_2^2)$ *such that one of the two tangles has an unknotted component.*

In this theorem, a "2-string essential free tangle decomposition" means a decomposition of a knot into two 2-string essential free tangles, where a 2-string essential free tangle is a pair $(B, t^1 \cup t^2)$ of a 3-ball B and a pair of arcs $t^1 \cup t^2$ properly embedded in B, such that t^1 and t^2 cannot be separated in B and the exterior of $t^1 \cup t^2$ in B is a genus two handlebody.

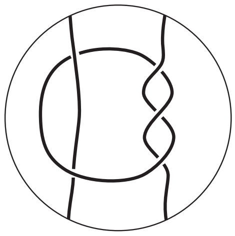

Figure 12: a 2-string essential free tangle with an unkntted component

Hence, by Figure 4, 8_{16} has a 2-string essential free tangle decomposition such that one of the two tangles has an unknotted component. This means that $t(8_{16}) = 2$ and $t(8_{16} \# K') = 2$ for any 2-bridge knot K'. Since the knots up to 7 crossings ara all 2-bridge knot, we see that 8_{16} is the first tunnel number two knot with tunnel number degeneration.

By the way, we here introduce a concept of "meridionally primitive". Let K be a knot with the tunnel number t. Then there is a genus $t+1$ Heegaard splitting (V_1, V_2) of S^3 such that V_1 contains K as a central loop of a handle.

Definition 3.6 We say that K is *meridionall primitive* if there is a genus $t+1$ Heegaard splitting (V_1, V_2) as above such that there is a meridian disk D_1 of V_1 and a meridian disk D_2 of V_2 with $D_1 \cap K = 1$ point and $D_1 \cap D_2 = \partial D_1 \cap \partial D_2 = 1$ point.

Then, concerning Q1, we had gotten the following :

Theorem 3.7 ([M1, M4]) *Let* K_1 *and* K_2 *be two knots with* $t(K_1) = t(K_2) = 1$. *Then* $t(K_1 \# K_2) = 3$ *if and only if none of* K_1 *and* K_2 *is meridionally primitive.*

We proved Theorem 3.3 by using this theorem. In fact, we proved that the knot K_m is not a meridionally primitive via Quantum ivariant formula due to Yokota ([Y]).

As a generalyzation of Theorem 3.7, we have ;

Theomre 3.8 ([M4]) *Let* K_1 *and* K_2 *be two knots in* S^1 *and suppose both* K_1 *and* K_2 *are meridionally small. Then* $t(K_1 \# K_2) = t(K_1) + t(K_2) + 1$ *if and only if none of* K_1 *and* K_2 *is meridionally primitive.*

In this theorem, we say that a knot K is *meridionally small* if the exterior $E(K) = cl(S^3 - N(K))$ contains no properly embedded essential surface F such that each component of ∂F is a meridian of K. By using the concept "meridionall small", we got the following :

Theorem 3.9 ([M5]) *Let $K_1, K_2, \cdots, K_n$ be all meridionally small knots in S^3. Then we have $t(K_1 \# K_2 \# \cdots \# K_n) \geq t(K_1) + t(K_2) + \cdots + t(K_n)$.*

Even if we drop the assumption of "meridionally small", Scharlemann and Schultens got the following :

Theorem 3.10 ([SS1]) *Let $K_1, K_2, \cdots, K_n$ be knots in S^3. Then we have $t(K_1 \# K_2 \# \cdots \# K_n) \geq n$.*

By the way, in studying of 3-manifolds, it is very important if a given 3-manifold contains an essential torus or not. As such studying for knot exteriors, we have characterized the knot types of tunnel number one knots containing essential tori in [MS].

4. Degeneration ratio

In the studying of the degeneration of tunnel numers, it seems that the ratio of $t(k_1 \# K_2)$ to $t(K_1) + t(K_2)$ is more important than the difference between $t(k_1 \# K_2)$ and $t(K_1) + t(K_2)$. Therefore Schrlemann and Schultens introduced *the degeneration ratio* $d(K_1, K_2)$ for any two knots K_1 and K_2 as follows :

$$d(K_1, K_2) = 1 - \frac{t(K_1 \# K_2)}{t(K_1) + t(K_2)}$$

Our first example "2+1=2" has the degeneration ration $\frac{1}{3}$ as follows :

$$d(K_1, K_2) = 1 - \frac{2}{2+1} = \frac{1}{3}$$

Then we can ask :

Q3 : What is the upper limit of the degeneration ratio ?

Concerning this question, Scharlemann and Schultens got :

Thoerem 4.1 ([SS2])] *For any two prime knots K_1 and K_2, we have $d(K_1, K_2) \leq \dfrac{3}{5}$.*

Recently, as the next step to "2+1=2", Nogueira got :

Theorem 4.2 ([Ng]) *There are infinitely many pairs of knot K_1 and K_2 such that*

$t(K_1) = 3$, $t(K_2) = 2$ *and* $t(K_1 \# K_2) = 3$, *i.e.*, "$3 + 2 = 3''$.

In this therem, Nogueira have gotton the concrete examples by using the knot K_m in Figure 10. Then the degeneration ratio is $\frac{2}{5}$ as follows :

$$d(K_1, K_2) = 1 - \frac{3}{3+2} = \frac{2}{5}$$

For the time being, this example is the biggest degeneration ratio. Thus, as the sequence of the series, we can ask :

Q4 : Are there infinitely many pairs of knot K_1, K_2 such that "4+3=4", "5+4=5", "6+5=6", "7+6=7", $\cdots$?

If there is such a sequence, we have the sequence of degeneration ratio $\frac{3}{7}$, $\frac{4}{9}$, $\frac{5}{11}$, $\frac{6}{13}$, $\frac{7}{15}$, $\cdots \to \frac{1}{2}$.

Concerning the degeneration ratio, we have several examples and results in [M6].

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