On Heegaard splittings of knot exteriors with tunnel number degenerations

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Abstract

Let K_1, K_2 be two knots with $t(K_1)+t(K_2) > 2$ and $t(K_1 \# K_2) = 2$. Then, in the present paper, we will show that any genus three Heegaard splittings of $E(K_1 \# K_2)$ is strongly irreducible and that $E(K_1 \# K_2)$ has at most four genus three Heegaard splittings up to homeomorphism. Moreover, we will give a complete classification of those four genus three Heegaard splittings and describe unknotting tunnel systems of knots $K_1 \# K_2$ corresponding to those Heegaard splittings.

Keywords: Heegaard splitting, unknotting tunnel system
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1. Introduction

Let K be a knot in S^3 and $t(K)$ the tunnel number of K, where $t(K)$ is the minimal number of arcs properly embedded in the exterior $E(K)$ whose complementary space is homeomorphic to a handlebody. By the definition of the tunnel number, we have $t(K) = g(E(K)) - 1$, where $g(E(K))$ is the Heegaard genus of $E(K)$.

Let K_1 and K_2 be two knots in S^3 and $K_1 \# K_2$ the connected sum of K_1 and K_2. Then, on the degeneration problem of tunnel numbers, i.e., the problem that if there are knots K_1 and K_2 with $t(K_1 \# K_2) < t(K_1) + t(K_2)$ or not, our first result is the following:

Theorem 1 ([4]). *There are infinitely many pairs of knots K_1 and K_2 such that $t(K_1) = 1$, $t(K_2) = 2$ and $t(K_1 \# K_2) = 2$.*

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We say that a knot K has a 2-string essential free tangle decomposition if (S^3, K) is decomposed into $(B_1, K \cap B_1)$ and $(B_2, K \cap B_2)$ such that $(B_i, K \cap B_i)$ is a 2-string essential tangle and $cl(B_i - N(K \cap B_i))$ is a genus two handlebody for both $i = 1, 2$, where $N(\cdot)$ denotes a regular neighborhood. Then we have characterized those knots in Theorem 1 as follows:

Theorem 2 ([5]). *Let K_1 and K_2 be two knots in S^3. Then we have :*
(1) *If $t(K_1) + t(K_2) > 2$ and $t(K_1 \# K_2) = 2$, then $t(K_1) + t(K_2) = 3$.*
(2) *$t(K_1) = 1, t(K_2) = 2$ and $t(K_1 \# K_2) = 2$ if and only if K_1 is a 2-bridge knot and K_2 is a knot with a 2-string essential free tangle decomposition such that at least one of the two tangles has an unknotted component.*

In the present paper, we investigate genus three Heegaard splittings of such knot exteriors $E(K_1 \# K_2)$ in Theorem 2(2) and describe unknotting tunnel systems of $K_1 \# K_2$ corresponding to those Heegaard splittings. First we will show:

Theorem 3. *Let K be a tunnel number two knot in S^3. Suppose a genus three Heegaard splitting of $E(K)$ is weakly reducible, then $E(K)$ is obtained from $E(K_1)$ and the exterior $E_V(K_2)$ of K_2 in V by gluing $\partial E(K_1)$ and ∂V, where K_1 is a tunnel number one knot in S^3 and K_2 is a tunnel number one knot in a solid torus V.*

Then we get:

Corollary 1. *Let K_1 and K_2 be two knots in Theorem 2(2). Then any genus three Heegaard splitting of $E(K_1 \# K_2)$ is strongly irreducible.*

Remark 1. In [3], it has been shown by Moriah that genus three Heegaard splittings of $E(K_1 \# K_2)$ are strongly irreducible for some subfamily of those knots K_1, K_2 in Theorem 2(2).

Next we have:

Theorem 4. *Let K_1 and K_2 be two knots in Theorem 2(2). Then $E(K_1 \# K_2)$ has at most four genus three Heegaard splittings up to homeomorphism.*

To give a complete classification of those four genus three Heegaard splittings in Theorem 4, we assume :

K_1 is a 2-bridge knot $S(\alpha, \beta)$ (Schubert's notation in [10]).

K_2 has a 2-string essential free tangle decomposition such that:

$(S^3, K_2) = (C_1, K_2 \cap C_1) \cup (C_2, K_2 \cap C_2)$ and
C_1 contains an unknotted component.

To state the classification theorem, we put the following cases:
Case 1: C_2 contains no unknotted component.
Case 2: C_2 contains an unkontted component.
Furthermore, we divide Case 2 into the following two sub-cases:
Case 2a: there is a self-homeomorphism of (S^3, K_2) exchanging the two tangles $(C_1, K_2 \cap C_1)$ and $(C_2, K_2 \cap C_2)$.
Case 2b: there is no self-homeomorphism of (S^3, K_2) exchanging the two tangles $(C_1, K_2 \cap C_1)$ and $(C_2, K_2 \cap C_2)$.
Then we get:

Theorem 5. *Let K_1 and K_2 be two knots in Theorem 2(2). Then we have the following complete classification of genus three Heegaard splittings of $E(K_1 \# K_2)$ up to homeomorphism, where n is the number of homeomorphism classes.*

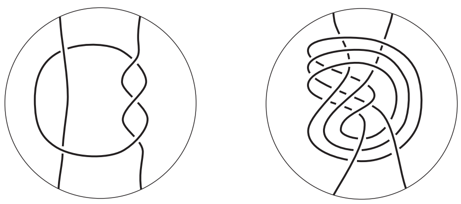
$$\textit{Case } 1 \begin{cases} n = 1 & \text{if } \beta \equiv \pm 1 \pmod{\alpha} \\ n = 2 & \text{if } \beta \not\equiv \pm 1 \pmod{\alpha} \end{cases}$$

$$\textit{Case } 2\text{a} \begin{cases} n = 1 & \text{if } \beta \equiv \pm 1 \pmod{\alpha} \\ n = 2 & \text{if } \beta \not\equiv \pm 1 \pmod{\alpha} \end{cases}$$

$$\textit{Case } 2\text{b} \begin{cases} n = 2 & \text{if } \beta \equiv \pm 1 \pmod{\alpha} \\ n = 4 & \text{if } \beta \not\equiv \pm 1 \pmod{\alpha} \end{cases}$$

Remark 2. The condition $\beta \equiv \pm 1 \pmod{\alpha}$ is equivalent to that K_1 is a torus knot.

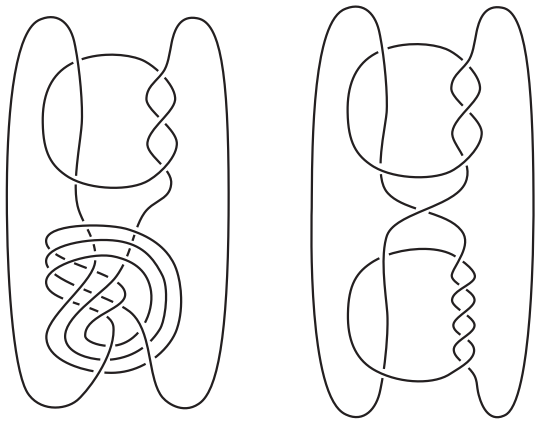
Example 1. In Figure 1, (i) is a 2-string essential free tangle with an unknotted component, and (ii) is a 2-string essential free tangle witn no unknotted component.



(i) (ii)

Figure 1: 2-string essential free tangles

Example 2. In Figure 2, (i) is a knot which has a 2-string essential free tangle decomposition such that one of the tangles has an unknotted component, and (ii) is a knot which has a 2-string essential free tangle decomposition such that both tangles have unknotted components, i.e., (i) is in Case 1 and (ii) is in Case 2 of Theorem 5.



(i) Case 1 (ii) Case 2

Figure 2: Knots with 2-string essential free tangle decompositions

Example 3. The knot illustrated in Figure 3 is Case 1 of Theorem 5 and the 2-bridge knot is of type (23, 7), i.e., $\beta \not\equiv \pm 1 \pmod{\alpha}$. Thus the knot exterior of the composite knot has two genus three Heegaard splittings, and the corresponding unknotting tunnel systems are $\{\tau_1, \tau_2\}$ and $\{\sigma_1, \sigma_2\}$ indicated in the figure.

Example 4. The two knots illustrated in Figure 4 are the same knots, because by sliding the 2-bridge knot along a sub-arc of the given knot, we can get the right-hand side knot from the left-hand side knot. Thus, since this case is Case 2b of Theorem 5, the knot exterior of the knot has four genus three Heegaard splittings and the corresponding unknotting tunnel systems are $\{\tau_1, \tau_2\}$, $\{\sigma_1, \sigma_2\}$, $\{\rho_1, \rho_2\}$ and $\{\delta_1, \delta_2\}$ indicated in the figure.

2. Proofs of Theorem 3 and Corollary 1

Let K be a knot in S^3, $N(K)$ a regular neighborhood of K in S^3 and $E(K) = cl(S^3 - N(K))$ the exterior. Put $H_1 \cup H_2$ be a Heegaard splitting of $E(K)$, where H_1 is a compression body and H_2 is a handlebody, i.e.,

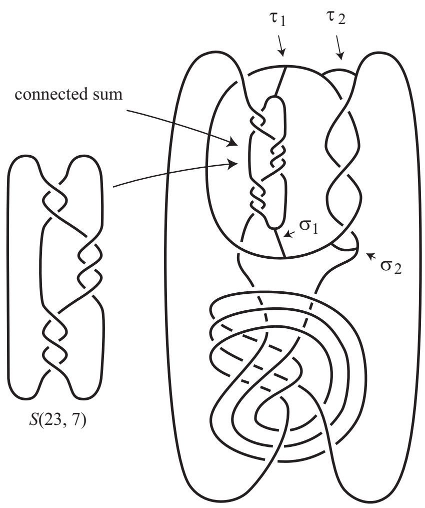


Figure 3: The two unknotting tunnel systems in Case 1

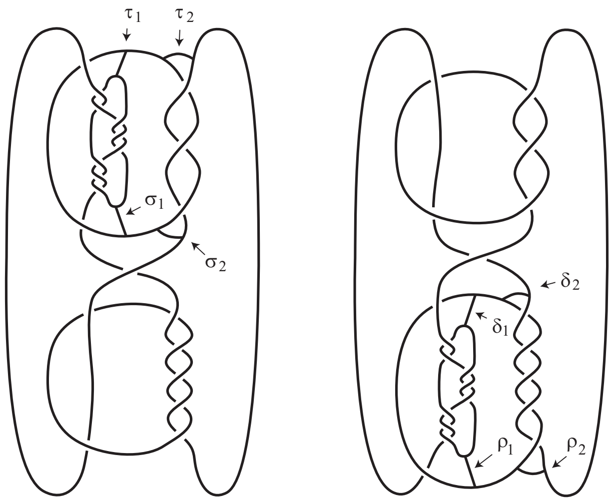


Figure 4: The four unknotting tunnel systems in Case 2b

$\partial E(K) \subset \partial H_1$. We say that the Heegaard splitting (H_1, H_2) is weakly reducible if there is an essential disk, say D_i, properly embedded in H_i $(i = 1, 2)$ such that $D_1 \cap D_2 = \emptyset$, and that (H_1, H_2) is strongly irreducible if it is not weakly reducible. For the definition of compression body, we refer [1], and the notion of weak reducibility and strong irreducibility of Heegaard splittings is also due to [1].

Let V be a solid torus and K a knot in $intV$. Let $N_V(K)$ be a regular neighborhood of K in V and $E_V(K) = cl(V - N_V(K))$ the exterior. We say that K is a tunnel number one knot in V if there is an arc γ properly embedded in $E_V(K)$ with $\gamma \cap \partial N_V(K) \neq \emptyset$ such that $cl(E_V(K) - N(\partial N_V(K) \cup \gamma))$ is a genus two handlebody (if $\gamma \cap \partial V \neq \emptyset$) or a genus two compression body (if $\partial\gamma \subset \partial N_V(K)$).

Proof of Theorem 3. Let $H_1 \cup H_2 = E(K)$ be a weakly reducible genus three Heegaard splitting with $\partial E(K) = \partial_- H_1$, and $D_1 \subset H_1$ and $D_2 \subset H_2$ be essential disks with $D_1 \cap D_2 = \emptyset$. Then we have the following three cases.

Case 1 : Both D_1 and D_2 are non-separating in H_1 and in H_2 respectively.

Put $H_1' = cl(H_1 - N(D_1))$, $H_2' = cl(H_2 - N(D_2))$, and put $V_1 = cl(H_1' - N(\partial H_1' - \partial E(K)))$, $V_2 = N(\partial H_1' - \partial E(K)) \cup N(D_2)$, $W_1 = N(\partial H_2') \cup N(D_1)$ and $W_2 = cl(H_2' - N(\partial H_2'))$ as illustrated in Figure 5.

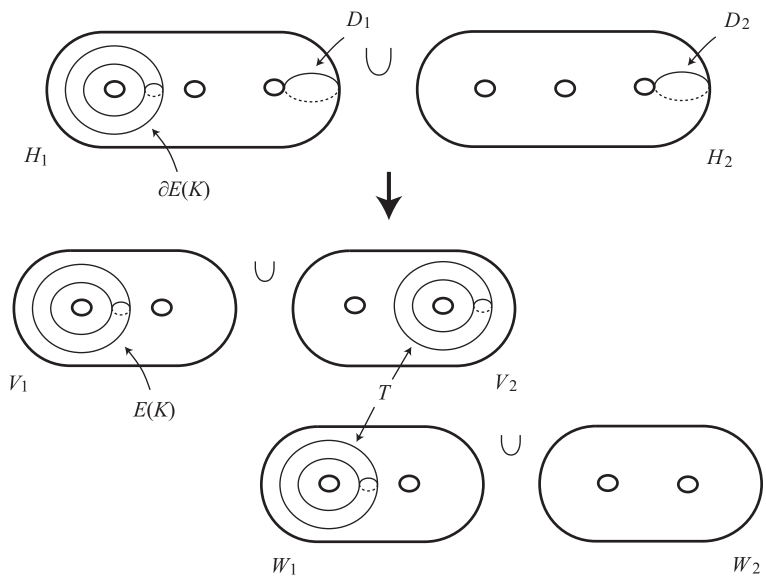


Figure 5: D_1 and D_2 are non-separating.

Put $T = V_2 \cap W_1$. If T consists of two tori (the case when ∂D_2 is separating in $\partial H_1'$), then each of the two tori is non-separating in S^3, a contradiction. Hence T is an incompressible torus in $E(K)$ and (H_1, H_2) is an amalgamation of (V_1, V_2) and (W_1, W_2) via T. By the solid torus theorem, T is a boundary of a solid torus, say U, in the $S^3 = E(K) \cup N(K)$, and $N(K)$ is contained in the sorid torus. Hence $W_1 \cup W_2$ is a knot exterior of some tunnel number one knot in S^3 because (W_1, W_2) is a genus two Heegaard splitting. In addition, $V_1 \cup V_2$ is a knot extrior of some tunnel number one knot in the solid torus U because (V_1, V_2) is a genus two Heegaard splitting.

Case 2 : Both D_1 and D_2 are separating in H_1 and in H_2 respectively. Let P_i be the once punctured torus bounded by ∂D_i in ∂H_i $(i = 1, 2)$. If $P_1 \cap P_2 \neq \emptyset$, then since $\partial D_1 \cap \partial D_2 = \emptyset$, we have $P_1 \subset P_2$ or $P_2 \subset P_1$. Then by some isotopy, we may assume that $P_1 = P_2$ and $\partial D_1 = \partial D_2$. Then $D_1 \cup D_2$ is a 2-sphere which bounds a 3-ball in $E(K)$. Then the knot K is a trivial knot or a tunnel number one knot, and this is a contradiction.

Hence $P_1 \cap P_2 = \emptyset$. Let $T_i = P_i \cup D_i$ be a torus in H_i $(i = 1, 2)$. If T_1 bounds a solid torus in H_1, then we can take a meridian disk in the solid torus, and we can take a meridian disk in the solid torus bounded by T_2 in H_2. Then this case is reduced to Case 1.

Suppose T_1 bounds $S^1 \times S^1 \times [0, 1]$, denoted by X, in H_1, and T_2 bounds a solid torus Y in H_2. Put $H_1' = cl(H_1 - X)$, $H_2' = cl(H_2 - Y)$, and put $V_1 = cl(H_2' - N(\partial H_2'))$, $V_2 = N(\partial H_2') \cup X$, $W_1 = N(\partial H_1') \cup Y$ and $W_2 = cl(H_1' - N(\partial H_1'))$ as illustrated in Figure 6.

Then by the reason similar to the proof of Case I, we see that $W_1 \cup W_2$ is a tunnel number one knot exterior in S^3, and $V_1 \cup V_2$ is a tunnel number one knot exterior in a solid torus.

Case 3 : One of D_1 and D_2 is separating and the other is non-separating.

Suppose D_1 is separating in H_1 and D_2 is non-separating in H_2. Since $\partial D_1 \cap \partial D_2 = \emptyset$, we can take a loop ℓ in $\partial H_1 = \partial H_2$ such that $\ell \cap \partial D_1 = \emptyset$ and $\ell \cap \partial D_2$ is a single point. Take a regular neighborhood of $D_2 \cup \ell$ in H_2, then it is a solid torus in H_2 and let D_2' be the frontier of the solid torus in H_2. Then D_2' is a separating essential disk in H_2 with $\partial D_1 \cap \partial D_2' = \emptyset$. Next suppose D_1 is non-separating in H_1 and D_2 is separating in H_2. Then similarly as above, we can take a separating disk D_1' in H_1 with $\partial D_1' \cap \partial D_2 = \emptyset$. Hence Case 3 is reduced to Case 2, and this completes the proof of Theorem 3. $\square$

Proof of Corollary 1. Put $K = K_1 \# K_2$ and suppose $E(K)$ has a genus three weakly reducible Heegaard splitting. Then by Theorem 3, there is an essential torus T in $E(K)$ which divides $E(K)$ into a tunnel number one knot

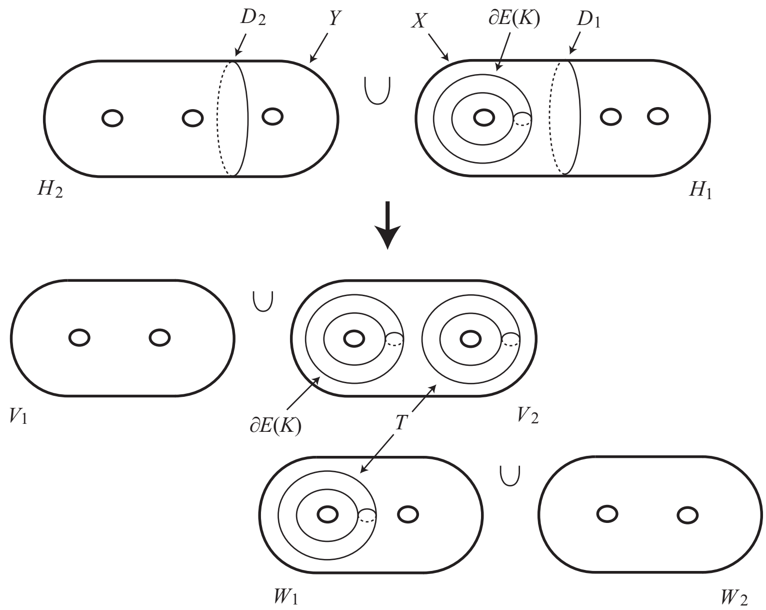


Figure 6: D_1 and D_2 are separating.

exterior $E(K_1')$ in S^3 and a tunnel number one knot exterior $E_V(K_2')$ in a solid torus V.

Suppose T is a swallow follow torus of the connected sum. Then, since $t(K_1) = 1$ and $t(K_2) = 2$, $E(K_1)$ is homeomorphic to $E(K_1')$ and $E(K_2)$ is homeomorphic to $E_V(K_2') \cup V'$ for some solid torus V'. This shows that $E(K_2)$ has a genus two Heegaard splitting and $t(K_2) = 1$. This is a contradiction, and T is not a swallow follow torus.

Let A be the decomposing annulus properly embedded in $E(K)$ corresponding to the connected sum of K.

First suppose $T \cap A = \emptyset$.

If $T \subset E(K_1)$, then since T is not a swallow follow torus, T is an essential torus in $E(K_1)$. But 2-bridge knot exterior contains no essential torus by [11]. This is a contradiction. If $T \subset E(K_2)$, then by the same reason as above, T is an essential torus in $E(K_2)$. But by [8, Theorem 1.2 and Lemma 1.3] or by [6, Proposition 2.1], this is a contradiction.

Hence $T \cap A \neq \emptyset$. Then, since we may assume that each component of $T \cap A$ is an essential loop in both T and A, we can take an essential annulus properly embedded in the 2-bridge knot exterior $E(K_1)$ whose boundary components are meridian loops. But this is a contradiction because 2-bridge knots are prime. After all, these contradictions show that $E(K)$ has no genus three weakly reducible Heegaard splitting, and this completes the proof of Corollary 1. $\square$

3. Proof of Theorem 4

Put $K = K_1 \# K_2$, and let $H_1 \cup H_2 = S^3$ be a genus three Heegaard splitting such that H_1 contains a knot K as a central loop of a handle of H_1. Let S be a decomposing 2-sphere of the connected sum $K_1 \# K_2$. Then by [5], we may assume that $S \cap H_1$ consists of two non-separating disks D_1, D_2 each of which intersects K in a single point and a non-separating annulus A. Similarly we may assume that $S \cap H_2$ consists of two non-separating annuli A_1 and A_2 (Figure 7).

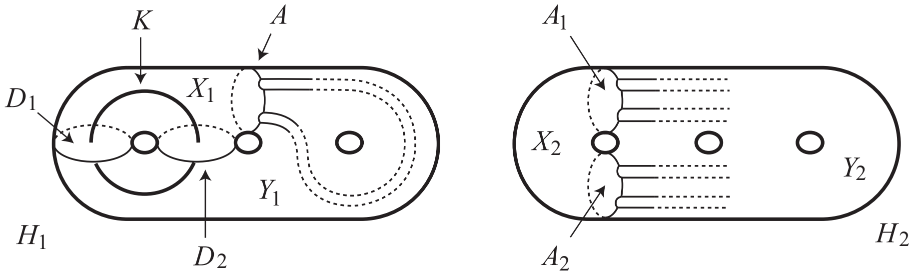


Figure 7: Heegaard splitting (H_1, H_2)

Then, S splits H_1 into two solid tori X_1 and Y_1 indicated in Figure 7, and S splits H_2 into two genus two handlebodies X_2 and Y_2 indicated in Figure 7. Put $I_1 = [0,1], I_2 = [1,2], I_3 = [2,3]$ and $I = I_1 \cup I_2 \cup I_3$. Let D_x (resp. D_y) be a disk and P_x (resp. P_y) be the central point of D_x (resp.D_y).

Put $B_1 = X_1 \cup_{(A=\partial D_x \times I_2)} (D_x \times I_2)$ and $B_2 = X_2 \cup_{(A_1 \cup A_2)=(\partial D_x \times (I_1 \cup I_3))} (D_x \times (I_1 \cup I_3))$. Then, since A and $A_1 \cup A_2$ are primitive annuli in ∂X_1 and in ∂X_2 respectively, B_1 and B_2 are two 3-balls and (B_1, B_2) gives a 2-bridge decomposition of the knot $K_1 = (B_1 \cap K) \cup (\mathrm{P}_x \times I)$ in the 3-sphere $B_1 \cup B_2$ (Figure 8). On the other hand, put $C_1 = Y_1 \cup_{(A=\partial D_y \times I_2)} (D_y \times I_2)$ and $C_2 = Y_2 \cup_{(A_1 \cup A_2)=(\partial D_y \times (I_1 \cup I_3))} (D_y \times (I_1 \cup I_3))$. Then, the arguments in the proof of the main theorem of [5] show that both C_1 and C_2 are 3-balls, and (C_1, C_2) gives a 2-string essential free tangle decomposition of the knot $K_2 = (C_1 \cap K) \cup (\mathrm{P}_y \times I)$ in the 3-sphere $C_1 \cup C_2$. We note that $\mathrm{P}_x \times I_2$ is an unknotted component in C_1 (Figure 8).

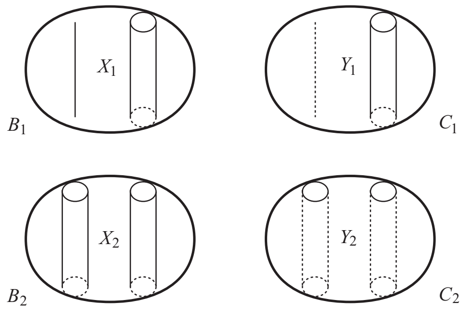


Figure 8: Tangle decompositions (B_1, B_2) and (C_1, C_2)

By the above arguments, we can see that any genus three Heegaard splitting of $E(K)$ is obtained from a 2-bridge decomposition of K_1 and a 2-string essential free tangle decomposition of K_2 by gluing $\partial(D_x \times I) = \partial(X_1 \cup X_2)$ and $\partial(D_y \times I) = \partial(Y_1 \cup Y_2)$. Then, by the uniqueness of prime decomposition of knots ([9]), by the uniqueness of 2-bridge decomposition ([10]), and by the uniqueness of 2-string essential free tangle decomposition ([8]), we have at most four choices of genus three Heegaard splittings up to homeomorphism, i.e., exchanging of B_1 and B_2 and exchanging of C_1 and C_2. See (i) $\sim$ (iv) of Figure 9. We note that X'_1, X'_2, Y'_1 and Y'_2 in Figure 9 are the other components of Heegaard splittings of $E(K)$ (c.f. Figure 10). Then, by $2 \times 2 = 4$, we complete the proof of Theorem 4. $\square$

4. Proof of Theorem 5

As we saw the proof of Theorem 4, genus three Heegaard splittings of $E(K)$ are dependent on the choice of 2-bridge decomposition of K_1 and free tangle decomposition of K_2.

Suppose we are in Case 1. Then, since C_2 contains no unknotted component, we have two Heegaard splittings (H_1, H_2) and (H'_1, H'_2) such that $H_1 = X_1 \cup Y_1$, $H_2 = X_2 \cup Y_2$, $H'_1 = X'_1 \cup Y_1$, $H'_2 = X'_2 \cup Y_2$, where (X_1, X_2) corresponds to (B_1, B_2), (X'_1, X'_2) corresponds to (B_2, B_1) and (Y_1, Y_2) corresponds to (C_1, C_2). See (i) and (ii) of Figures 9 and 10.

If $\beta \equiv \pm 1 \pmod{\alpha}$, then by [7, Theorem 5.2] there is an isotopy of $S^3 = B_1 \cup B_2$ which sends B_1 to B_2 and leaves K_1 invariant. This implies that X'_1 is isotopic to X_1 and X'_2 is isotopic to X_2. Thus (H_1, H_2) is isotopic to (H'_1, H'_2) and we have $n = 1$.

Suppose $\beta \not\equiv \pm 1 \pmod{\alpha}$, and suppose (H'_1, H'_2) is homeomorphic to (H_1, H_2). Then the homeomorphism between (H'_1, H'_2) and (H_1, H_2) takes

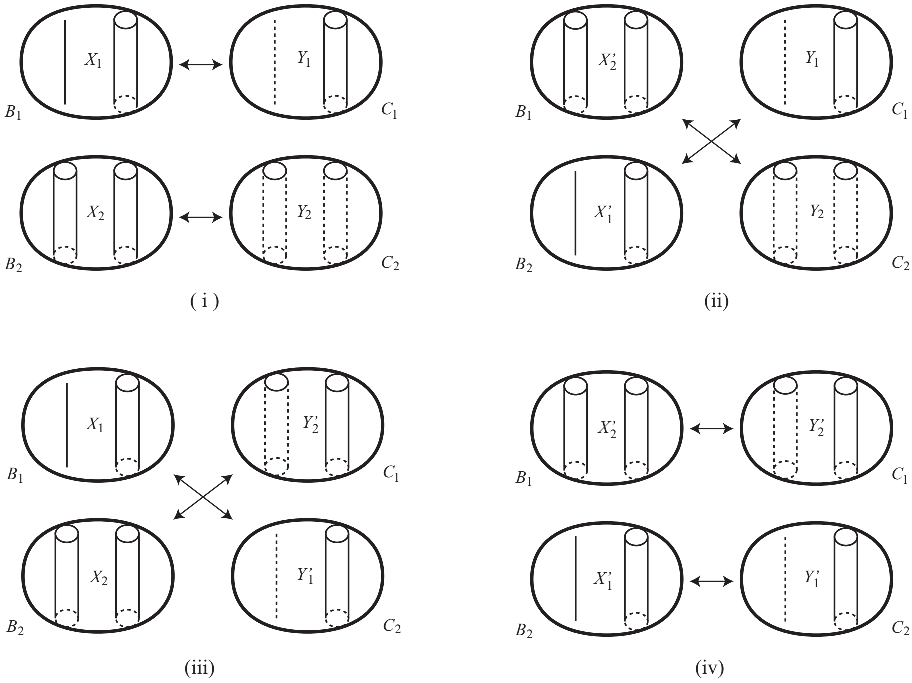


Figure 9: Four combinations

(X_1', X_2') to (X_1, X_2), and this homeomorphism induces a self-homeomorphism of $S^3 = B_1 \cup B_2$ which exchanges B_1 and B_2 and leaves K_1 invariant. Then, since $\beta \not\equiv \pm 1 \pmod{\alpha}$ and by [7, Theorem 5.2], this homeomorphism reverses the orientation of the 2-bridge knot K_1, and this shows that the homeomorphism between (H_1', H_2') and (H_1, H_2) exchanges A_1 and A_2. This means that there is a self-homeomorphism of Y_2 which exchanges A_1 and A_2.

Let a_1 and a_2 be the central loops of A_1 and A_2 respectively. Then we can regard $(Y_2, a_1 \cup a_2)$ is a genus two Heegaard diagram of S^3 because Y_2 is a complementary space of a 2-string free tangle and a_1 and a_2 are the central loops of the 2-handles. Then, by taking a complete meridian disk system of the genus two handlebody Y_2, we have $\pi_1(Y_2) \cong < x, y \mid - >$, where x and y correspond to those meridian disks. Then by a_1 and a_2, we have words w_1 and w_2 in the letters x and y, and we have $\pi_1(S^3) \cong < x, y \mid w_1, w_2 >$. Then, by [2], the representation of $\pi_1(S^3)$ can be deformed into a standard one by a sequence of mutual substitutions. However, this is impossible because w_1 and w_2 have the same lengths by the existence of a self-homeomorphism of Y_2 exchanging a_1 and a_2. This contradiction shows that (H_1', H_2') is not

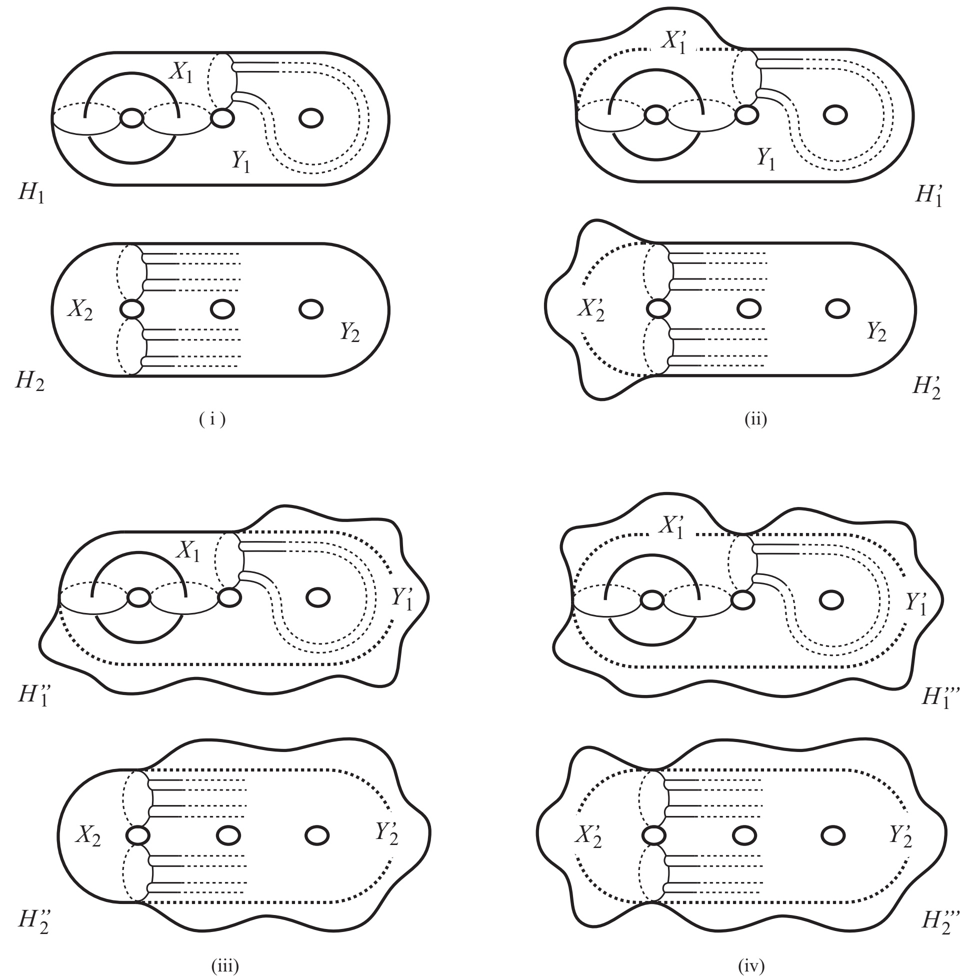


Figure 10: Four Heegaard splittings

homeomorphic to (H_1, H_2), and shows that $n = 2$.

Next, suppose we are in Case 2. In this case, since C_2 also has an unknotted component, We have four Heegaard splittings (H_1, H_2), (H_1', H_2'), (H_1'', H_2'') and (H_1''', H_2''') such that $H_1 = X_1 \cup Y_1$, $H_2 = X_2 \cup Y_2$, $H_1' = X_1' \cup Y_1$, $H_2' = X_2' \cup Y_2$, $H_1'' = X_1 \cup Y_1'$, $H_2'' = X_2 \cup Y_2'$ and $H_1''' = X_1' \cup Y_1'$, $H_2''' = X_2' \cup Y_2'$, where (X_1, X_2) corresponds to (B_1, B_2) and (X_1', X_2') corresponds to (B_2, B_1), (Y_1, Y_2) corresponds to (C_1, C_2) and (Y_1', Y_2') corresponds to (C_2, C_1). See (i), (ii), (iii) and (iv) of Figures 9 and 10.

If we are in Case 2b, then, since there is no homeomorphism exchanging C_1 and C_2, the situation is similar to Case 1 and we see that (iii) and (iv) are not homeomorphic to (i) or (ii). This shows that $n = 2$ if $\beta \equiv \pm 1 \pmod{\alpha}$ and $n = 4$ if $\beta \not\equiv \pm 1 \pmod{\alpha}$.

Suppose we are in Case 2a. Then, since there is a homeomorphism exchanging C_1 and C_2, we have a homeomorphism which takes $Y_1' \cup Y_2'$ to $Y_1 \cup Y_2$ leaving $Y_1 \cap K = Y_1' \cap K$ invariant. This homeomorphism induces a self-homeomorphism on $A_1 \cup A_2$ and on $A \cup D_1 \cup D_2$. Then, since any 2-bridge knot is strongly invertible, this homeomorphism extends to a homeomorphism from $X_1' \cup X_2'$ to $X_1 \cup X_2$ which leaves $X_1 \cap K = X_1' \cap K$ invariant. Thus, this case is reduced to Case 1, and we have $n = 1$ if $\beta \equiv \pm 1 \pmod{\alpha}$ and $n = 2$ if $\beta \not\equiv \pm 1 \pmod{\alpha}$. This completes the proof of Theorem 5. $\square$

5. Unknotting tunnel systems

In the present section, we will describe the unknotting tunnel systems corresponding to those Heegaard splittings of Theorem 5. Recall the Heegaard splitting (H_1, H_2) and consider the unknotting tunnel system $\{\tau_1, \tau_2\}$ in H_1 as in Figure 11. Then τ_1 is divided by S into two arcs $\tau_1' \cup \tau_1''$. Then τ_1' is an upper or a lower tunnel of the 2-bridge knot K_1, τ_1'' is an arc in C_1 connecting $K_2 \cap C_1$ and A, and τ_2 is a core loop of the solid torus Y_1 together with a sub-arc of K_2. Then, by applying these situations to the knots K_1 and K_2 as illustrated in Figures 3 and 4, we have those unknotting tunnel systems illustrated in Figures 3 and 4. In fact, by the deformation (i) $\sim$ (iv) as in Figure 12, we see that τ_2 is in the position of Figure 3.

At the end of the present paper, we put the following problem:

Problem *Classify the genus three Heegaard splittings of $E(K_1 \# K_2)$ up to isotopy.*

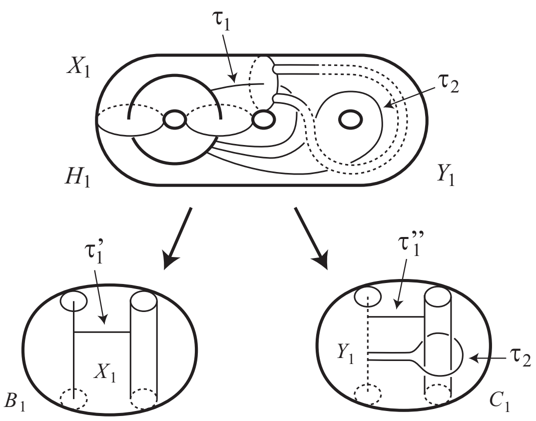


Figure 11: Heegaard splitting and unknotting tunnel system

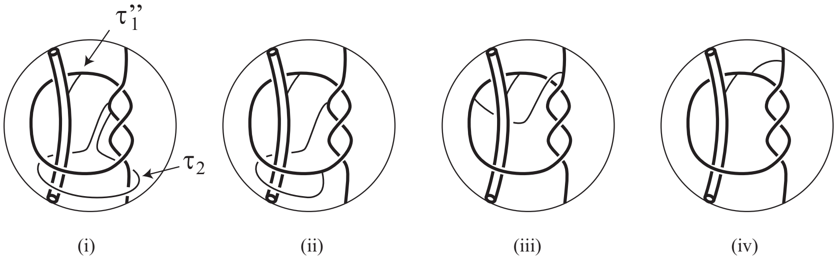


Figure 12: Deformation of τ_2

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