Essential tori in 3-string free tangle decompsitions of knots

by

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Dedicated to Professor Cameron McA. Gordon for his 60th birthday

Abstract. In the present paper, we characterize those knot types which have 3-string essential free tangle decompositions and have essential tori in the exteriors.

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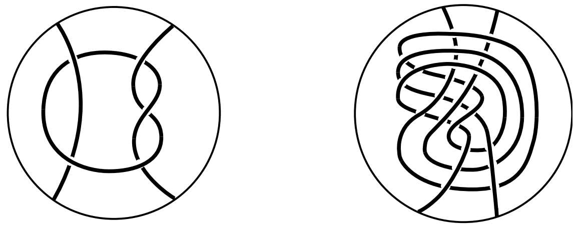
1. Introduction

Let K be a knot in the 3-sphere S^3, $N(K)$ a regular neighborhood of K and $E(K) = cl(S^3 - N(K))$ the exterior. Concerning relation between 2-string free tangle decompositions of knots and essential tori in the exteriors, we know the following.

Suppose K has a 2-string free tangle decomposition, then (1) if both tangles of the tangle decomposition are essential, then $E(K)$ contains no essential torus by [Oz] (c.f. Proposition 2.1 in the present paper), (2) if one of the two tangles is inessential, then K has tunnel number one and those knots which have essential tori in $E(K)$ have been characterized in [MS], (3) if both tangles of the tangle decomposition are inessential, then K is a 2-bridge knot and $E(K)$ conatins no essential torus by [Sc].

In the present paper, we study the case when K has a 3-string free tangle decomposition. Let B be a 3-ball and $t^1 \cup t^2 \cup \cdots \cup t^n$ n-arcs properly embedded in B, then we call the pair $(B, t^1 \cup t^2 \cup \cdots \cup t^n)$ an *n-string tangle*. We say that $(B, t^1 \cup t^2 \cup \cdots \cup t^n)$ is *essential* if $cl(\partial B - N(t^1 \cup t^2 \cup \cdots \cup t^n))$ is incompressible in $cl(B - N(t^1 \cup t^2 \cup \cdots \cup t^n))$ if $n > 1$ and t^1 is a knotted arc in B if $n = 1$, where $N(t^1 \cup t^2 \cup \cdots \cup t^n)$ is a regular neighborhood of $t^1 \cup t^2 \cup \cdots \cup t^n$ in B, and it is *inessential* if it is not essential. We say that $(B, t^1 \cup t^2 \cup \cdots \cup t^n)$ is *trivial* if it is homeomorophic to $(D^2 \times I, \{x_1, x_2, \cdots, x_n\} \times I)$, where D^2 is a 2-disk, I is the unit interval and $x_1, x_2, \cdots, x_n$ are n points in $\text{int}D^2$, that a component t^i is *unknotted* if (B, t^i) is homeomorphic to a 1-string trivial tangle. Finally, according as [Ko], we say that $(B, t^1 \cup t^2 \cup \cdots \cup t^n)$ is *free* if $cl(B - N(t^1 \cup t^2 \cup \cdots \cup t^n))$ is homeomorphic to a



(1) (2)

Figure 1

genus n handlebody.

Remark 1 A 2-string free tangle is inessential if and only if it is a trivial tangle.

We say that a knot K in the 3-sphere S^3 has an *n-string free tangle decomposition* if (S^3, K) is decomposed into two n-string free tangles $(B_1, t_1^1 \cup t_1^2 \cup \cdots \cup t_1^n) \cup (B_2, t_2^1 \cup t_2^2 \cup \cdots \cup t_2^n)$. The decomposition is *essential* if both tangles are essential, and the decomposition is *inessential* if it is not essential. Then we will show the following.

Theorem 1.1 *Let K be a knot which has a 3-string essential free tangle decomposition. Then $E(K)$ contains an essential torus if and only if K is a connected sum of two knots K_1 and K_2 both of which have 2-string essential free tangle decompositions $(S^3, K_1) = (B_1, t_1^1 \cup t_1^2) \cup (B_2, t_2^1 \cup t_2^2)$ and $(S^3, K_2) = (C_1, s_1^1 \cup s_1^2) \cup (C_2, s_2^1 \cup s_2^2)$ such that at least two of these four tangles have unknotted components.*

Moreover, any essential torus in $E(K)$ is a swallow-follow torus in the connected sum, because the connected sum $K = K_1 \# K_2$ is the prime decomposition of K.

Typical examples of a 2-string essential free tangle with an unknotted component and a 2-string essential free tangle with no unknotted component are illustrated in Figure 1(1), (2). We note that if the two arcs of a 2-string free tangle are both unknotted components, then it is a trivial tangle by [Go].

Remark 2 (1) A 3-bridge knot has an essential torus in the exterior if and only if it is a connected sum of two 2-bridge knots by [Sc]. (2) Since a knot K has tunnel number two if and only if K has a 3-string free tangle decomposition with at least one trivial tangle, to characterize those knots which have 3-string free tangle decompositions and contain essential tori in the exteriors, as the first step, we need to characterize tunnel number two knots which contain essential tori in the exteriors.

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Thrughout the present paper, we will work in the piecewise linear category. For a manifold X and a subcomplex Y in X, we denote a regular neighborhood of Y in X by $N(Y, X)$ or $N(Y)$ simply.

2. Proof of Theorem 1.1

First we consider the 2-string case.

Proposition 2.1 ([Oz, Theorem 1.2, Lemma 1.3]) *Let K be a knot which has a 2-string essential free tangle decomposition, then $E(K)$ contains no essential torus.*

Before the proof, we prepare several lemmata.

Lemma 2.2 *Let V be a genus n handlebody and A a separating incompressible annulus properly embedded in V. Then A splits V into two handlebodies V_1 and V_2 such that (1) $g(V_1)+g(V_2) = n+1$ and (2) A is primitive in at least one of V_1 and V_2, where $g(\cdot)$ is the genus of the handlebody and A is primitive if there is a disk properly embedded in the handlebody which intersects A in an essential arc of A.*

Proof. If A is ∂-parallel, then A splits V into a solid torus and a genus n handlebody, and A is primitive in the solid torus. Suppose A is not ∂-parallel. Then A is a union of a separating disk, say D, and a band, say b. Since D splits V into two handlebodies W_1 and W_2 with $g(W_1)+g(W_2) = n$, the band b is contained in W_1 or in W_2, say W_2. Then we can put $V_1 = W_1 \cup (b \times I)$, $V_2 = cl(W_2 - (b \times I))$. Thus $g(V_1)+g(V_2) = n+1$ and A is primitive in V_1. This completes the proof of the lemma. □

Lemma 2.3 *There is no local knot in any free tangle.*

Proof. Let $(B, t^1 \cup t^2 \cup \cdots \cup t^n)$ be a free tangle, and suppose it has a local knot. Then there is a 2-sphere S in B which bounds the local knot. Put $A = cl(S - N(t^1 \cup t^2 \cup \cdots \cup t^n))$, then A is a separating incompressible annulus in the handlebody $cl(B - N(t^1 \cup t^2 \cup \cdots \cup t^n))$. Then by Lemma 2.2, A splits the handlebody into two handelbodies, but one of them is a non-trivial knot exterior, a contradiction. □

Lemma 2.4 *Let $(B, t^1 \cup t^2)$ be a 2-string essential free tangle, and let A be an incompressible and not ∂-parallel annulus properly embedded in $B - (t^1 \cup t^2)$. Then A is properly isotopic in $B - (t^1 \cup t^2)$ to the annulus $cl(\partial N(t^i) - \partial B)$ for $i = 1$ or 2.*

Proof. Let C be a 3-ball in B cut off by A. If $(t^1 \cup t^2) \subset C$, then $cl(B - C)$ is a knot exterior in S^3. Then by Lemma 2.2, $cl(B - C)$ is a solid torus and A is ∂-parallel, a contradiction. Hence we may assume that $t^1 \subset C$ and $t^2 \cap C = \emptyset$. Then by Lemma

2.3, we get the conclusion. □

Lemma 2.5 *Let $(B, t^1 \cup t^2 \cup t^3)$ be a 3-string essential free tangle, and let A be an incompressible and not ∂-parallel annulus properly embedded in $B - (t^1 \cup t^2 \cup t^3)$. Then one of the following (1), (2) and (3) holds.*

(1) A is properly isotopic in $B - (t^1 \cup t^2 \cup t^3)$ to the annulus $cl(\partial N(t^i) - \partial B)$ for $i = 1, 2$ or 3,

(2) A cuts off a 3-ball from B, say C, with $C \cap \partial B = D_1 \cup D_2$ (two disks) such that C contains two components of t^1, t^2, t^3, say $t^1 \cup t^2$, $D_1 \cap (t^1 \cup t^2)$ is one point (or three points) and $D_2 \cap (t_1 \cup t_2)$ is three points (or one point respectively),

(3) A cuts off a 3-ball from B, say C, with $C \cap \partial B = D_1 \cup D_2$ (two disks) such that C contains two components of t^1, t^2, t^3, say $t^1 \cup t^2$, both $D_1 \cap (t^1 \cup t^2)$ and $D_2 \cap (t_1 \cup t_2)$ consist of two points.

Proof. Let C be a 3-ball in B cut off by A. If $(t^1 \cup t^2 \cup t^3) \subset C$, then we have a contradiction similarly to the proof of Lemma 2.4. If one component of t^1, t^2, t^3 is contained in C, then we get the conclusion (1) similarly to the proof of Lemma 2.4. If two components of t^1, t^2, t^3 are contained in C, then by the incompressibility of A in $B - (t^1 \cup t^2 \cup t^3)$, we get the conclusion (2) or (3).

Lemma 2.6 *Let $(B, t^1 \cup t^2 \cup t^3)$ be a 3-string essential free tangle, and let D be a disk properly embedded in B with $D \cap (t^1 \cup t^2 \cup t^3) = D \cap t^3 =$ a point. Suppose D splits t^3 into two arcs t^3_1, t^3_2, and splits B into two 3-balls B_1, B_2 with $t^1 \subset B_1$ and $t^2 \subset B_2$. Then both tangles $(B_1, t^1 \cup t^3_1)$ and $(B_2, t^2 \cup t^3_2)$ are 2-string essential free tangles and at least one of the two tangles has an unknotted component.*

Proof. Put $A = cl(D - N(t^3))$. Then A is a separating incompressible and not ∂-parallel annulus properly embedded in $cl(B - N(t^1 \cup t^2 \cup t^3))$. Then, since $cl(B - N(t^1 \cup t^2 \cup t^3))$ is a genus three handlebody and by Lemma 2.2, both $cl(B_1 - N(t^1 \cup t^3_1))$ and $cl(B_2 - N(t^2 \cup t^3_2))$ are genus two handlebodies. Hence both $(B_1, t^1 \cup t^3_1)$ and $(B_2, t^2 \cup t^3_2)$ are 2-string free tangles. If at least one of $(B_1, t^1 \cup t^3_1)$ and $(B_2, t^2 \cup t^3_2)$ is inessential, say $(B_1, t^1 \cup t^3_1)$, then there is a compressing disk for $cl(\partial B_1 - N(t^1 \cup t^3_1))$, say D_1, properly embedded in $cl(B_1 - N(t^1 \cup t^3_1))$ which separates t^1 and t^3_1. Since, D_1 intersects A in only inessential arcs of A, we can eliminate the intersections by some isotopies and we get a compressing disk for $cl(\partial B - N(t^1 \cup t^2 \cup t^3))$, a contradiction. Hence both $(B_1, t^1 \cup t^3_1)$ and $(B_2, t^2 \cup t^3_2)$ are essential.

By Lemma 2.2, A is primitive in $cl(B_1 - N(t^1 \cup t^3_1))$ or in $cl(B_2 - N(t^2 \cup t^3_2))$, say $cl(B_1 - N(t^1 \cup t^3_1))$. Then the annulus $cl(B_2 - N(t^1 \cup t^3_1)) \cap N(t^3_1)$ is primitive in $cl(B_1 - N(t^1 \cup t^3_1))$. Hence $cl(B_1 - N(t^1))$ is a solid torus because $N(t^3_1)$ is a 2-handle for $cl(B_1 - N(t^1 \cup t^3_1))$. This shows that t^1 is an unknotted arc in B_1. This completes the proof of the lemma. □

Proof of Proposition 2.1. Let $(S^3, K) = (B_1, t_1^1 \cup t_1^2) \cup (B_2, t_2^1 \cup t_2^2)$ be a 2-string essential free tangle decomposition, and let T be an essential torus in $E(K)$. Then by the incompressibility in $E(K)$ of $\partial B_1 - (t_1^1 \cup t_1^2) = \partial B_2 - (t_2^1 \cup t_2^2)$, say F, T intersects F in essential loops in T. Then, we may assume that T intersects $B_i (i = 1, 2)$ in incompressible annuli in $B_i - (t_i^1 \cup t_i^2)$. Moreover, by taking $\#(T \cap B_1) = \#(T \cap B_2)$ to be minimal, we may assume that each component of $T \cap B_i$ is not ∂-parallel in $B_i - (t_i^1 \cup t_i^2)(i = 1, 2)$. Then by Lemma 2.4, T is isotopic to the torus $\partial N(K)$ in $E(K)$. This contradicts the essentiality of T in $E(K)$, and completes the proof of the proposition. □

Remark 3 Let L be a 2-component link which has a 2-string essential free tangle decomposition, then by the same argument as the proof of Proposition 2.1, we see that $E(L)$ contains no essential torus.

Proof of Theorem 1.1. Let $(S^3, K) = (B_1, t_1^1 \cup t_1^2 \cup t_1^3) \cup (B_2, t_2^1 \cup t_2^2 \cup t_2^3)$ be a 3-string essential free tangle decomposition, and let T be an essential torus in $E(K)$. Then we may assume that $T \cap B_1 = A_1 \cup A_3 \cup \cdots \cup A_{2n-1}$, $T \cap B_2 = A_2 \cup A_4 \cup \cdots \cup A_{2n}$, where $A_i (i = 1, 3, \cdots, 2n-1)$ is an incompressible and not ∂-parallel annulus properly embedded in $B_1 - (t_1^1 \cup t_1^2 \cup t_1^3)$ and $A_j (j = 2, 4, \cdots, 2n)$ is an incompressible and not ∂-parallel annulus properly embedded in $B_2 - (t_2^1 \cup t_2^2 \cup t_2^3)$. Then A_i and A_j satisfy the conclusions of Lemma 2.5. If some A_i in B_1 satisfies the conclusion (3) of Lemma 2.5, then A_{i+1} in B_2 satisfies the conclusion (3) of Lemma 2.5 and $A_i \cup A_{i+1}$ is a torus in $E(K)$. This means K has more than one component, a contradiction. Hence all A_i and A_j satisfy the conclusions (1) or (2) of Lemma 2.5. If all annuli satisfy the conclusion (1) of Lemma 2.5, then T is isotopic to the torus $\partial N(K)$ in $E(K)$, a contradiction. Hence we may assume that A_1 satisfies the conclusion (2) of Lemma 2.5.

Let C_1 be the 3-ball cut off by A_1 in B_1, and put $C_1 \cap \partial B_1 = D_1^1 \cup D_1^2 =$ two disks such that $D_1^1 \cap (t_1^1 \cup t_1^2) =$ a point of ∂t_1^1 and $D_1^2 \cap (t_1^1 \cup t_1^2) = \partial t_1^2 \cup$ a point of ∂t_1^1. Let C_2 be the 3-ball in B_2 cut off by A_2 and put $C_2 \cap \partial B_2 = D_2^1 \cup D_2^2 =$ two disks. Then we may assume that $D_1^2 = D_2^1$, $t_2^1 \cup t_2^2 \subset C_2$, $D_2^1 \cap (t_2^1 \cup t_2^2) = \partial t_2^1 \cup$ a point of ∂t_2^2 and $D_2^2 \cap (t_2^1 \cup t_2^2) =$ a point of ∂t_2^2. A schematic picture is illustrated in Figure 2.

Then A_3 and A_4 satisfy the conclusion (1) of Lemma 2.5, and hence $T = A_1 \cup A_2 \cup A_3 \cup A_4$. Let D be a disk in B_1 such that $D \cap A_3 = \partial D$ and $D \cap t_1^3 =$ a point as indicated in Figure 2. Perform a 2-surgery for T along D, then we get a 2-sphere intersecting K in two points. Then by sliding the 2-sphere along K, we get a 2-sphere S such that $S \cap K =$ two points, $S \cap B_1 =$ a disk, say S_1, and $S \cap B_2 =$ a disk, say S_2. Then by Lemma 2.6, $S_i (i = 1, 2)$ splits $(B_i, t_i^1 \cup t_i^2 \cup t_i^3)$ into two 2-sting essential

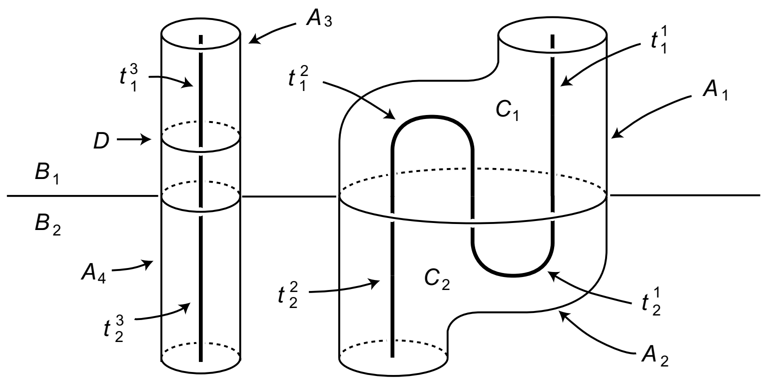

Figure 2

free tangles such that at least one of the two tangles has an unknotted component. Moreover by Proposition 2.1, the connected sum is the prime decomposition of K. This completes the proof of " if part " of Theorem 1.1.

Conversely, suppose $K = K_1 \# K_2$, $(S^3, K_1) = (B_1, t_1^1 \cup t_1^2) \cup (B_2, t_2^1 \cup t_2^2)$ and $(S^3, K_2) = (C_1, s_1^1 \cup s_1^2) \cup (C_2, s_2^1 \cup s_2^2)$ are 2-string essential free tangle decompositions such that at least two of these four tangles have unknotted components. Then by changing the letters if necessary, we may assume that at least one of $(B_1, t_1^1 \cup t_1^2)$ and $(C_1, s_1^1 \cup s_1^2)$ has an unknotted component and at least one of $(B_2, t_2^1 \cup t_2^2)$ and $(C_2, s_2^1 \cup s_2^2)$ has an unknotted component. Then by tracing back the above arguments, we see that K has a 3-string essential free tangle decomposition and $E(K)$ contains an essential torus, i.e., the swallow-follow torus. This completes the proof of Theorem 1.1. □

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