Essential tori in 3-string free tangle decompositions of links

by

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Abstract. In the present paper, we characterize those link types which have 3-string essential free tangle decompositions and have essential tori in their exteriors.

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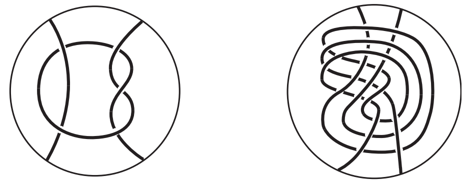
1. Introduction

Let L be a knot or a link in the 3-sphere S^3, $N(L)$ a regular neighborhood of L and $E(L) = cl(S^3 - N(L))$ its exterior. We say that a torus in $E(L)$ is *essential* if it is incompressible and not ∂-parallel. Concerning relation between free tangle decompositions of L and essential tori in their exteriors, in the previous paper [1], we studied the case when L is a knot, and we have shown the following.

Theorem 1.1([1, Theorem 1.1] *Let K be a knot which has a 3-string essential free tangle decomposition. Then $E(K)$ contains an essential torus if and only if K is a connected sum of two knots K_1 and K_2 both of which have 2-string essential free tangle decompositions $(S^3, K_1) = (B_1, t_1^1 \cup t_1^2) \cup (B_2, t_2^1 \cup t_2^2)$ and $(S^3, K_2) = (C_1, s_1^1 \cup s_1^2) \cup (C_2, s_2^1 \cup s_2^2)$ such that at least two of these four tangles have unknotted components. In this case, any essential torus in $E(K)$ is a swallow-follow torus in the connected sum.*

Remark 1 Let K be a knot which has a 3-bridge decomposition. Then, by [2], $E(K)$ contains an essential torus if and only if K is a connected sum of two 2-bridge knots.

Typical examples of a 2-string essential free tangle with an unknotted component and a 2-string essential free tangle with no unknotted component are illustrated in Figure 1. We note that if the two arcs of a 2-string free tangle are both unknotted components, then it is a trivial tangle by [3].



(1) (2)

Figure 1

Suppose a knot or a link L has a 2-string free tangle decomposition. Then (1) if both tangles of the tangle decomposition are essential, then $E(L)$ contains no essential torus by [4] or by [1, Proposition 2.1 or Remark 3], (2) if one of the two tangles is inessential, then L has tunnel number one and those links which have essential tori in $E(L)$ have been characterized in [5] or in [6], (3) if both tangles of the tangle decomposition are inessential, then L is a 2-bridge knot or link and $E(L)$ contains no essential torus by [2].

In the present paper, we study the case when a link L has a 3-string free tangle decomposition. Let B be a 3-ball and $t^1 \cup t^2 \cup \cdots \cup t^n$ n-arcs properly embedded in B, then we call the pair $(B, t^1 \cup t^2 \cup \cdots \cup t^n)$ an *n-string tangle*. We say that $(B, t^1 \cup t^2 \cup \cdots \cup t^n)$ is *essential* if $cl(\partial B - N(t^1 \cup t^2 \cup \cdots \cup t^n))$ is incompressible in $cl(B - N(t^1 \cup t^2 \cup \cdots \cup t^n))$ if $n > 1$ and t^1 is a knotted arc in B if $n = 1$, where $N(t^1 \cup t^2 \cup \cdots \cup t^n)$ is a regular neighborhood of $t^1 \cup t^2 \cup \cdots \cup t^n$ in B, and it is *inessential* if it is not essential. We say that $(B, t^1 \cup t^2 \cup \cdots \cup t^n)$ is *trivial* if it is homeomorophic to $(D^2 \times I, \{x_1, x_2, \cdots, x_n\} \times I)$, where D^2 is a 2-disk, I is the unit interval and $x_1, x_2, \cdots, x_n$ are n points in $\mathrm{int}D^2$, that a component t^i is *unknotted* if (B, t^i) is homeomorphic to a 1-string trivial tangle. Finally, we say that $(B, t^1 \cup t^2 \cup \cdots \cup t^n)$ is *free* if $cl(B - N(t^1 \cup t^2 \cup \cdots \cup t^n))$ is homeomorphic to a genus n handlebody.

We say that a link L in the 3-sphere S^3 has an *n-string free tangle decomposition* if (S^3, L) is decomposed into two n-string free tangles $(B_1, t_1^1 \cup t_1^2 \cup \cdots \cup t_1^n) \cup (B_2, t_2^1 \cup t_2^2 \cup \cdots \cup t_2^n)$. The decomposition is *essential* if both tangles are essential, and the decomposition is *inessential* if it is not essential.

Remark 2 A 2-string free tangle is inessential if and only if it is a trivial tangle.

Let $V = D^2 \times S^1$ be a solid torus, and consider a knot or a link L in V. Then we say that L in V has an *n-string tangle decomposition* if (V, L) is decomposed into two n-string tangles $(D^2 \times [0, \frac{1}{2}], t_1^1 \cup t_1^2 \cup \cdots \cup t_1^n) \cup (D^2 \times [\frac{1}{2}, 1], t_2^1 \cup t_2^2 \cup \cdots \cup t_2^n)$, where we regard S^1 as the quotient space $[0, 1]/\{0\} = \{1\}$. The decomposition is *free* if both tangles are free, the decomoposition is *essential* if both tangles are essential, the decomposition is *inessential* if it is not essential, and the tangle decompostion is an *n-bridge decomposition* if both tangles are homeomophic to an n-string trivial tangles. We say that an n-string tangle $(D^2 \times I, t^1 \cup t^2 \cup \cdots \cup t^n)$ with $\partial(t^1 \cup t^2 \cup \cdots \cup t^n) \subset D^2 \times \{0,1\}$ is an *n-string braid* if it is homeomorphic to the product space $(D^2, \{x_1, x_2, \cdots, x_n\}) \times I$ by a homeomorphism which takes $D^2 \times \{0,1\}$ to $D^2 \times \{0,1\}$. Then we say that a knot or a link L in V is an *n-string closed braid* if L has an n-string tangle decomposition with both n-string braid. Under these terms and notations, we will show the following.

Theorem 1.2 *Let L be a link with at least two components. Suppose L has a 3-bridge decomposition. Then $E(L)$ contains an essential torus if and only if one of the following conclusions (1), (2) holds :*

(1) L is a connected sum of two 2-bridge knots or links L_1 and L_2, where at least one of L_1 and L_2 is a link. In this case, any essential torus in $E(L)$ is a swallow follow torus in the connected sum.

(2) Let $L_1 = K_1 \cup K_2$ be a 2-bridge link except for a trivial link or a Hopf link, and let L_2 be a knot or a link in the solid torus $V = D^2 \times S^1$ which has a 2-bridge decomposition such that any meridian disk of V intersects L_2 in at least two points. Let $N(K_2)$ be a regular neighborhood of K_2 in S^3, and let $h : V \to N(K_2)$ be a homeomorphism. Then L is the link $K_1 \cup h(L_2)$. In this case, if L is not a $(3, 3n)$-torus link for any $n > 1$ then any essential torus in $E(L)$ is isotopic to the glueing torus $\partial N(K_2) = h(\partial V)$, and if L is a $(3, 3n)$-torus link for some $n > 1$ then there are infinitely many non-isotopic essential tori in $E(L)$ which are all mutually homeomorphic. We note that L is a $(3, 3n)$-torus link if and only if L_1 is a $(2, 2n)$-torus link and L_2 is a 2-string closed braid in V with a suitable glueing map h.

Theorem 1.3 *Let L be a link with at lease two components. Suppose L has a 3-string essential free tangle decomposition. Then $E(L)$ contains an essential torus if and only if one of the following conclusions (1), (2) holds :*

(1) L is a connected sum of two knots or links L_1 and L_2 (at lease one of L_1 and L_2 is a link) both of which have 2-string essential free tangle decompositions $(S^3, L_1) = (B_1, t_1^1 \cup t_1^2) \cup (B_2, t_2^1 \cup t_2^2)$ and $(S^3, L_2) = (C_1, s_1^1 \cup s_1^2) \cup (C_2, s_2^1 \cup s_2^2)$ such that at least two of these four tangles have unknotted components. In this case, any essential torus in $E(L)$ is a swallow follow torus in the connected sum.

(2) Let $L_1 = K_1 \cup K_2$ be a 2-component link in S^3 which has a 2-string essential free tangle decomposition $(S^3, L_1) = (B_1, t_1^1 \cup t_1^2) \cup (B_2, t_2^1 \cup t_2^2)$ with $K_1 = t_1^1 \cup t_2^1$ and $K_2 = t_1^2 \cup t_2^2$, and let L_2 be a knot or a link in the solid torus $V = D^2 \times S^1$ which has a 2-string free tangle decomposition $(V, L_2) = (D^2 \times [0, \frac{1}{2}], s_1^1 \cup s_1^2) \cup (D^2 \times [\frac{1}{2}, 1], s_2^1 \cup s_2^2)$ such that any meridian disk of V intersects L_2 in at least two points. Let $N(K_2)$ be a regular neighborhood of K_2 in S^3, and let $h : V \to N(K_2)$ be a homeomorphism. Then

L is the link $K_1 \cup h(L_2)$ with at least one of the following three conditions, and in this case any essential torus in $E(L)$ is isotopic to the glueing torus $\partial N(K_2) = h(\partial V)$:
(i) *t_i^1 is an unknotted arc in B_i for both $i = 1, 2$,*
(ii) *L_2 is a 2-string closed braid in the solid torus V,*
(iii) *at least one of t_1^1, t_2^1 is an unknotted arc in B_1, B_2 respectively, and at least one of $(D^2 \times [0, \frac{1}{2}], s_1^1 \cup s_1^2)$ and $(D^2 \times [\frac{1}{2}, 1], s_2^1 \cup s_2^2)$ is a 2-string braid.*

Remark 3 Since a link L has tunnel number two if and only if L has a 3-string free tangle decomposition with at least one trivial tangle, to characterize those links which have 3-string free tangle decompositions and contain essential tori in their exteriors, as the first step, we need to characterize tunnel number two links which contain essential tori in their exteriors.

Throughout the present paper, we will work in the piecewise linear category. For a manifold X and a subcomplex Y in X, we denote a regular neighborhood of Y in X by $N(Y, X)$ or $N(Y)$ simply.

2. Proofs of Theorems 1.2 and 1.3

First we prove Theorem 1.3. But before the proof, we prepare four technical lemmata. We say that an annulus in the boundary of a handlebody is *primitive* if there is a non-separating disk properly embedded in the handlebody which intersects the annulus in an essential arc in the annulus.

Lemma 2.1 *Let V be a genus n handlebody and A a separating incompressible annulus properly embedded in V. Then A splits V into two handlebodies V_1 and V_2 such that (1) $g(V_1) + g(V_2) = n + 1$ and (2) A is primitive in at least one of V_1 and V_2, where $g(\cdot)$ is the genus of the handlebody.*

Proof. If A is ∂-parallel, then A splits V into a solid torus and a genus n handlebody, and A is primitive in the solid torus. Suppose A is not ∂-parallel. Then A is a union of a separating disk, say D, and a band, say b. Since D splits V into two handlebodies W_1 and W_2 with $g(W_1) + g(W_2) = n$, the band b is contained in W_1 or in W_2, say W_2. Then we can put $V_1 = W_1 \cup (b \times I)$, $V_2 = cl(W_2 - (b \times I))$. Thus $g(V_1) + g(V_2) = n + 1$ and A is primitive in V_1. This completes the proof of the lemma. □

Lemma 2.2 *Let $(B, t^1 \cup t^2 \cup t^3)$ be a 3-string essential free tangle, and let A be an incompressible and not ∂-parallel annulus properly embedded in $B - (t^1 \cup t^2 \cup t^3)$. Then one of the following (1), (2) and (3) holds.*
(1) *A is properly isotopic in $B - (t^1 \cup t^2 \cup t^3)$ to the annulus $cl(\partial N(t^i) - \partial B)$ for $i = 1, 2$ or 3,*

(2) *A cuts off a* 3*-ball from* B*, say* C*, with* $C \cap \partial B = D_1 \cup D_2$ *(two disks) such that* C *contains two components of* t^1, t^2, t^3*, say* $t^1 \cup t^2$*,* $D_1 \cap (t^1 \cup t^2)$ *is one point (or three points) and* $D_2 \cap (t_1 \cup t_2)$ *is three points (or one point respectively),*
(3) *A cuts off a* 3*-ball from* B*, say* C*, with* $C \cap \partial B = D_1 \cup D_2$ *(two disks) such that* C *contains two components of* t^1, t^2, t^3*, say* $t^1 \cup t^2$*, both* $D_1 \cap (t^1 \cup t^2)$ *and* $D_2 \cap (t_1 \cup t_2)$ *consist of two points.*

Proof. Let C be a 3-ball in B cut off by A. If $(t^1 \cup t^2 \cup t^3) \subset C$, then by Lemma 2.1 $cl(B - C)$ is a solid torus and A is primitive in the solid torus. Then A is ∂-parallel and we have a contradiction. If exactly one component of t^1, t^2, t^3 is contained in C, then by the incompressibility of A in $B - (t^1 \cup t^2 \cup t^3)$ and since $(B, t^1 \cup t^2 \cup t^3)$ has no local knot by Lemma 2.1, we get the conclusion (1). If exactly two components of t^1, t^2, t^3 are contained in C, then by the incompressibility of A in $B - (t^1 \cup t^2 \cup t^3)$, we get the conclusion (2) or (3). $\square$

Lemma 2.3 *Let* $(B, t^1 \cup t^2 \cup t^3)$ *be a* 3*-string essential free tangle, and let* D *be a disk properly embedded in* B *with* $D \cap (t^1 \cup t^2 \cup t^3) = D \cap t^3 =$ *a point. Suppose* D *splits* t^3 *into two arcs* t^3_1, t^3_2*, and splits* B *into two 3-balls* B_1, B_2 *with* $t^1 \cup t^3_1 \subset B_1$ *and* $t^2 \cup t^3_2 \subset B_2$*. Then both tangles* $(B_1, t^1 \cup t^3_1)$ *and* $(B_2, t^2 \cup t^3_2)$ *are* 2*-string essential free tangles and at least one of the two tangles has an unknotted component.*

Proof. Put $A = cl(D - N(t^3))$. Then A is a separating incompressible and not ∂-parallel annulus properly embedded in $cl(B - N(t^1 \cup t^2 \cup t^3))$. Then, since $cl(B - N(t^1 \cup t^2 \cup t^3))$ is a genus three handlebody and by Lemma 2.1, both $cl(B_1 - N(t^1 \cup t^3_1))$ and $cl(B_2 - N(t^2 \cup t^3_2))$ are genus two handlebodies. Hence both $(B_1, t^1 \cup t^3_1)$ and $(B_2, t^2 \cup t^3_2)$ are 2-string free tangles. If at least one of $(B_1, t^1 \cup t^3_1)$ and $(B_2, t^2 \cup t^3_2)$ is inessential, say $(B_1, t^1 \cup t^3_1)$, then there is a compressing disk for $cl(\partial B_1 - N(t^1 \cup t^3_1))$, say D_1, properly embedded in $cl(B_1 - N(t^1 \cup t^3_1))$ which separates t^1 and t^3_1. Since, D_1 intersects A only in inessential arcs of A, we can eliminate the intersections by some isotopies and we get a compressing disk for $cl(\partial B - N(t^1 \cup t^2 \cup t^3))$, a contradiction. Hence both $(B_1, t^1 \cup t^3_1)$ and $(B_2, t^2 \cup t^3_2)$ are essential.

By Lemma 2.1, A is primitive in $cl(B_1 - N(t^1 \cup t^3_1))$ or in $cl(B_2 - N(t^2 \cup t^3_2))$, say in $cl(B_1 - N(t^1 \cup t^3_1))$. Then the annulus $cl(B_2 - N(t^1 \cup t^3_1)) \cap N(t^3_1)$ is primitive in $cl(B_1 - N(t^1 \cup t^3_1))$. Hence $cl(B_1 - N(t^1))$ is a solid torus because $N(t^3_1)$ is a 2-handle for $cl(B_1 - N(t^1 \cup t^3_1))$. This shows that t^1 is an unknotted arc in B_1. This completes the proof of the lemma. $\square$

Let $(B, t^1 \cup t^2)$ be a 2-string free tangle, and let α be an arc in ∂B which connects a point of ∂t^1 and a point of ∂t^2. Let $D(\alpha)$ be a regular neighborhood of α in ∂B, and put $c = \partial D(\alpha)$ as in Figure 2.

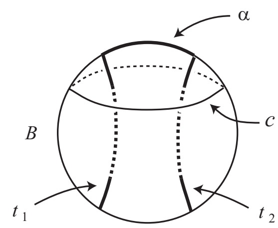


Figure 2

Lemma 2.4 *Suppose c is primitive in the genus two handlebody $cl(B - N(t^1 \cup t^2))$. Then $(B, t^1 \cup t^2)$ is homeomorphic to the product space $(D^2, \{x_1, x_2\}) \times I$ by a homeomorphism which takes $D(\alpha)$ to $D^2 \times \{0\}$.*

Proof. Let D be a 2-disk, and let $D \times I$ be a 2-handle for $cl(B - N(t^1 \cup t^2))$ along c. Then, since c is primitive, $cl(B - N(t^1 \cup t^2)) \cup D \times I$ is a solid torus, say V. Regard $D \times I$ as a 2-handle for the 3-ball B, and let R be the 3-ball bounded by $D(\alpha) \cup D \times \{0\}$ as indicated in Figure 3(1), where c is identified with $\partial D \times \{0\}$ and $\partial D \times \{1\} \cap D(\alpha) = \emptyset$.

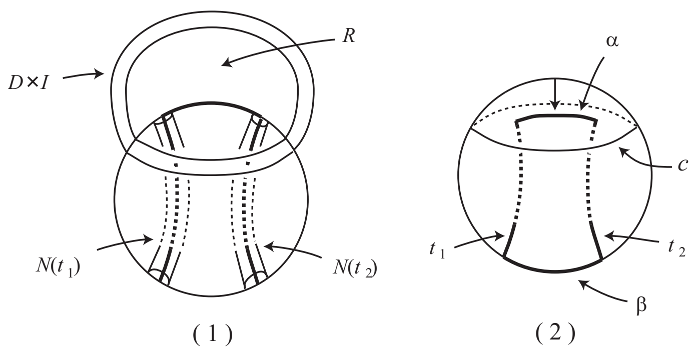


Figure 3

Put $N = N(t^1) \cup R \cup N(t^2)$. Then, since N can be regarded as a 2-handle for V and $V \cup N$ is a 3-ball, the co-core of the 2-handle N is a trivial arc in the 3-ball. Then, since we can regard $t^1 \cup \alpha \cup t^2$ as the co-core of the 2-handle N, we see that $t^1 \cup \alpha \cup t^2$ is a trivial arc in the 3-ball $V \cup N$.

Now, regard the 3-ball $V \cup N$ as the 3-ball B, then, by pushing $t^1 \cup \alpha \cup t^2$ into the interior of B slightly, we can regard $t^1 \cup \alpha \cup t^2$ as a trivial arc properly embedded in B as in Figure 3(2). Hence we can take a disk Δ in B with $\partial \Delta = (t^1 \cup \alpha \cup t^2) \cup \beta$ and $\beta \subset \partial B$. Then by standard innermost argument, we can deform Δ so that $D(\alpha) \cap \beta = \emptyset$. Then we can regard the disk Δ as the rectangle $t_1 \cup \alpha \cup t_2 \cup \beta$ as in Figure 3(2). Hence by pushing back α into ∂B, we see that $t_1 \cup t_2$ consists of two parallel arcs properly embedded in B. Then, since $(B, t_1 \cup t_2)$ is a free tangle, $(B, t_1 \cup t_2)$ is a trivial tangle and we can construct the product structure of $(B, t^1 \cup t^2)$ $\cong (D^2, \{x_1, x_2\}) \times I$ so that $D(\alpha)$ corresponds to $D^2 \times \{0\}$. $\square$

Proof of Theorem 1.3. Let $(S^3, L) = (B_1, t_1^1 \cup t_1^2 \cup t_1^3) \cup (B_2, t_2^1 \cup t_2^2 \cup t_2^3)$ be a 3-string essential free tangle decomposition, and let T be an essential torus in $E(L)$. Then we may assume that each component of $T \cap \partial B_1$ is a loop. If there is an inessential loop in T, then the disk bounded by the loop is a disk properly embedded in B_1 or in B_2. Then by the essentiality of the tangles, we can remove the disk. Thus we may assume that each component of $T \cap \partial B_1$ is an essential loop in T, and we can put $T \cap B_1 = A_1 \cup A_3 \cup \cdots \cup A_{2n-1}$, $T \cap B_2 = A_2 \cup A_4 \cup \cdots \cup A_{2n}$, where A_i $(i = 1, 3, \cdots, 2n-1)$ is an incompressible and not ∂-parallel annulus properly embedded in $B_1 - (t_1^1 \cup t_1^2 \cup t_1^3)$ and A_j $(j = 2, 4, \cdots, 2n)$ is an incompressible and not ∂-parallel annulus properly embedded in $B_2 - (t_2^1 \cup t_2^2 \cup t_2^3)$. Then A_i and A_j satisfy the conclusions of Lemma 2.2. If all annuli satisfy the conclusion (1) of Lemma 2.2, then T is isotopic to a component of $\partial N(L)$ in $E(L)$, a contradiction. Hence we may assume that A_1 satisfies the conclusion (2) or (3) of Lemma 2.2.

Case I : A_1 satisfies the conclusion (2). Let C_1 be the 3-ball cut off by A_1 in B_1, and suppose $t_1^1 \cup t_1^2 \subset C_1$. Put $C_1 \cap \partial B_1 = D_1^1 \cup D_1^2 =$ two disks such that $D_1^1 \cap (t_1^1 \cup t_1^2) =$ a point of ∂t_1^1 and $D_1^2 \cap (t_1^1 \cup t_1^2) = \partial t_1^2 \cup$ a point of ∂t_1^1 . Let C_2 be the 3-ball in B_2 cut off by A_2 and put $C_2 \cap \partial B_2 = D_2^1 \cup D_2^2 =$ two disks. Then we may assume that $D_1^2 = D_2^1$, $t_2^1 \cup t_2^2 \subset C_2$, $D_2^1 \cap (t_2^1 \cup t_2^2) = \partial t_2^1 \cup$ a point of ∂t_2^2 and $D_2^2 \cap (t_2^1 \cup t_2^2) =$ a point of ∂t_2^2 (c.f. Figure 4).

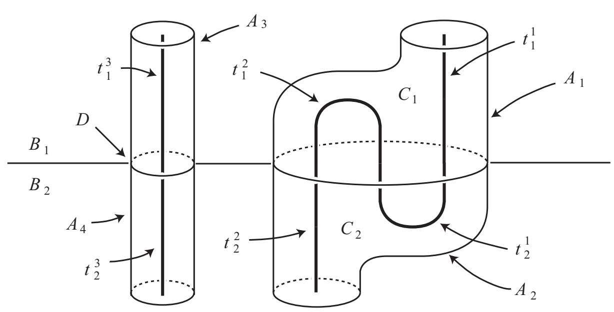


Figure 4

In this case, $t_1^1 \cap D_1^1$ is identified with $t_2^2 \cap D_2^2$ or with a point of ∂t_2^3. Suppose $t_1^1 \cap D_1^1$ is identified with $t_2^2 \cap D_2^2$. Then by noting Lemma 2.2, we have $T = A_1 \cup A_2$. Then, by performing a 2-surgery for T along $D_1^1 = D_2^2$ (i.e., a cut and paste operation along the 2-handle $D_1^1 \times I = D_2^2 \times I$), we get a 2-sphere S which gives a connected sum of L such that $S \cap B_1 =$ a disk, say S_1, and $S \cap B_2 =$ a disk, say S_2. Then by Lemma 2.3, S_i $(i = 1, 2)$ splits $(B_i, t_i^1 \cup t_i^2 \cup t_i^3)$ into two 2-string essential free tangles such that at least one of the two tangles has an unknotted component. Hence, at least two of the four tangles obtained by the above splitting have unknotted components. Thus we get the conclusion (1) of Theorem 1.3.

Next suppose $t_1^1 \cap D_1^1$ is identified with a component of ∂t_2^3. Then by noting Lemma 2.2, we have $T = A_1 \cup A_2 \cup A_3 \cup A_4$, and both A_3 and A_4 satisfy the conclusion (1) of Lemma 2.2. Let D be a disk in $\partial B_1 = \partial B_2$ such that $D \cap (A_3 \cup A_4) = \partial D$ and $D \cap (t_1^3 \cup t_2^3) =$ a point as indicated in Figure 4. Perform a 2-surgery for T along D, then we get a 2-sphere intersecting a component of L in two points. Then by sliding the 2-sphere along the component, we get a 2-sphere S which gives a connected sum of L. Then by the same argument as above, we get the conclusion (1) of Theorem 1.3.

Conversely, suppose $L = L_1 \# L_2$, $(S^3, L_1) = (B_1, t_1^1 \cup t_1^2) \cup (B_2, t_2^1 \cup t_2^2)$ and $(S^3, L_2) = (C_1, s_1^1 \cup s_1^2) \cup (C_2, s_2^1 \cup s_2^2)$ are 2-string essential free tangle decompositions such that at least two of these four tangles have unknotted components. Then by changing the letters if necessary, we may assume that at least one of $(B_1, t_1^1 \cup t_1^2)$ and $(C_1, s_1^1 \cup s_1^2)$ has an unknotted component and at least one of $(B_2, t_2^1 \cup t_2^2)$ and $(C_2, s_2^1 \cup s_2^2)$ has an unknotted component. Then by tracing back the above arguments, we see that L has a 3-string essential free tangle decomposition and $E(L)$ contains an essential torus, i.e., a swallow-follow torus. In this case, since knots or links with 2-string essential free tangle decompositions have no essential tori in their exteriors ([1], [4]), we see that any essential torus in $E(L)$ is a swallow follow torus in the connected sum.

Case II : A_1 satisfies the conclusion (3). Let C_1 be the 3-ball cut off by A_1 in B_1, and suppose $t_1^2 \cup t_1^3 \subset C_1$. Put $C_1 \cap \partial B_1 = D_1^1 \cup D_1^2 =$ two disks such that $D_1^1 \cap (t_1^2 \cup t_1^3) =$ two points of $\partial(t_1^2 \cup t_1^3)$ and $D_1^2 \cap (t_1^2 \cup t_1^3) =$ the other two points of $\partial(t_1^2 \cup t_1^3)$. Let C_2 be the 3-ball in B_2 cut off by A_2 and put $C_2 \cap \partial B_2 = D_2^1 \cup D_2^2 =$ two disks. Then, since $D_2^i \cap (t_2^1 \cup t_2^2 \cup t_2^3)$ consists of two points for both $i = 1, 2$ by Lemma 2.2, we may assume that $D_2^1 = D_1^2$, $t_2^2 \cup t_2^3 \subset C_2$, $D_2^1 \cap (t_2^2 \cup t_2^3) =$ two points of $\partial(t_2^2 \cup t_2^3)$ and $D_2^2 \cap (t_2^2 \cup t_2^3) =$ the other two points of $\partial(t_2^2 \cup t_2^3)$. Then $D_2^2 = D_1^1$ and we have $T = A_1 \cup A_2$.

Since A_1 is a separating incompressible annulus in the handlebody $cl(B_1 - N(t_1^1 \cup t_1^2 \cup t_1^3))$, by Lemma 2.1, A_1 is primitive in $cl(B_1 - C_1 - N(t_1^1))$ or in $cl(C_1 - N(t_1^2 \cup t_1^3))$. Similarly A_2 is primitive in $cl(B_2 - C_2 - N(t_2^1))$ or in $cl(C_2 - N(t_2^2 \cup t_2^3))$. Since $cl(C_i - N(t_i^2 \cup t_i^3))$ is a genus two handlebody for $i = 1, 2$ by Lemma 2.1, $(C_i, t_i^2 \cup t_i^3)$ is a free tangle. Then $(t_1^2 \cup t_1^3) \cup (t_2^2 \cup t_2^3)$ is a link in the solid torus $C_1 \cup C_2$, which has a 2-string free tangle decomposition. If A_i is compressible in $cl(C_i - N(t_i^2 \cup t_i^3))$, then the compressing disk extends to a compressing disk for $cl(\partial B_i - N(t_i^1 \cup t_i^2 \cup t_i^3))$, a contradiction. Hence A_i is incompressible in $cl(C_i - N(t_i^2 \cup t_i^3))$ for $i = 1, 2$, and $\partial(C_1 \cup C_2)$ is incompressible in $(C_1 \cup C_2) - (t_1^2 \cup t_1^3 \cup t_2^2 \cup t_2^3)$. Thus we put $L_2 = t_1^2 \cup t_1^3 \cup t_2^2 \cup t_2^3$ and $V = C_1 \cup C_2$.

Next for $i = 1, 2$, take a 2-disk D_i, a point x_i in $\text{int}D_i$ and an arc $t_i^4 = x_i \times I$ in $D_i \times I$. Put $M_i = cl(B_i - C_i) \cup D_i \times I$ by identifying A_i with $\partial D_i \times I$. Then $(M_i, t_i^1 \cup t_i^4)$ is a 2-string essential free tangle, because if $(M_i, t_i^1 \cup t_i^4)$ has a compressing disk, then it can be regarded as a compressing disk of $(B_i, N(t_i^1 \cup t_i^2 \cup t_i^3))$. Put $K_1 = t_1^1 \cup t_2^1$, $K_2 = t_1^4 \cup t_2^4$ and $L_1 = K_1 \cup K_2$. Then L_1 is a 2-component link in the 3-sphere $S^3 = M_1 \cup M_2$ which has a 2-string essential free tangle decomposition. Suppose A_i is primitive in $cl(B_i - C_i - N(t_i^1))$ for both $i = 1, 2$, then, by the arguments in the proof of Lemma 2.3, t_i^1 is an unknotted component in B_i for both $i = 1, 2$, and this is the condition (i) in (2) of Theorem 1.3. Next suppose A_i is primitive in $cl(C_i - N(t_i^2 \cup t_i^3))$ for both $i = 1, 2$. Then by Lemma 2.4, both tangles are homeomorphic to the product tangle, i.e., a 2-string braid. This means that L_2 in V is a 2-string closed braid, and we have the condition (ii) in (2) of Theorem 1.3. Finally, suppose the annulus is primitive in at least one of $cl(B_1 - C_1 - N(t_1^1))$ and $cl(B_2 - C_2 - N(t_2^1))$ and the annulus is primitive in at least one of $cl(C_1 - N(t_1^2 \cup t_1^3))$ and $cl(C_2 - N(t_2^2 \cup t_2^3))$. Then we have the condition (iii) in (2) of Theorem 1.3.

Conversely, if a link $L = L_1 \cup L_2$ satisfies at least one of the conditions (i), (ii), (iii) in (2) of Theorem 1.3, then by tracing back the above arguments, we see that L has a 3-string essential free tangle decomposition and $E(L)$ contains an essential torus, i.e., the glueing torus $\partial N(K_2) = h(\partial V)$. In this case, if there is another essential torus in $E(L)$ not isotopic to T then we may assume that the torus intersects T in essential loops and $E(L_1)$ contains an essential annulus. Then, by a little observation, L_1 is a $(2, n)$-torus link and this means that the tangle decomposition of L_1 is inessential, a contradiction. Hence any essential torus in $E(L)$ is isotopic to the glueing torus. This completes the proof of Theorem 1.3. □

Proof of Theorem 1.2. The proof of Theorem 1.2 is similar to that of Theorem 1.3. So we give only a brief sketch. Let $(S^3, L) = (B_1, t_1^1 \cup t_1^2 \cup t_1^3) \cup (B_2, t_2^1 \cup t_2^2 \cup t_2^3)$ be a 3-bridge decompositon of L, and let T be an essential torus in $E(L)$. Then, by the arguments in [2], we may assume that each component of $T \cap B_i$ $(i = 1, 2)$ is an annulus. Moreover, since $(B_i, t_i^1 \cup t_i^2 \cup t_i^3)$ is a trivial tangle, each annulus in $T \cap B_i$ is an unknotted annulus in B_i and divide B_i into a 3-ball containing some component(s) of $t_i^1 \cup t_i^2 \cup t_i^3$ and a solid torus containing the other component(s) of $t_i^1 \cup t_i^2 \cup t_i^3$. Then by arguments similar to those in the proof of Theorem 1.3, we have : $T = A_1 \cup A_2$ with $A_1 \subset B_1$ and $A_2 \subset B_2$, or $T = A_1 \cup A_2 \cup A_3 \cup A_4$ with $A_1 \cup A_3 \subset B_1$ and

$A_2 \cup A_4 \subset B_2$. Then by arguments similar to those in the proof of Theorem 1.3, we get the conclusions of Theorem 1.2. In particular, in the conclusion (2), L_1 cannot be a trivial link or a Hopf link because $\partial N(K_2)$ in $E(L_1)$ is incompressible and not parallel to the other component of $\partial E(L_1)$.

Conversely, if L satisfy the conclusion (1) or (2) of Theorem 1.2, then by tracing back the above arguments, we see that L has a 3-bridge decomposition and $E(L)$ contains an essential torus. More precisely speaking, in the conclusion (1), since 2-bridge knots or links have no essential tori by [6], any essential torus in $E(L)$ is a swallow follow torus in the connected sum. In the conclusion (2), suppose there is another essential torus T' in $E(L)$ not isotopic to the glueing torus T. Then we may assume that T' intersects $E(L_1)$ in essential annuli and intersects $cl(V - N(L_2))$ in essential annuli. Then, by a little observation, $E(L_1)$ is a $(2, 2n)$-torus link exterior for some $n > 1$, and $cl(V - N(L_2))$ is homeomorphic to (a disk with two holes) $\times S^1$. Then, by the glueing map which extends these Seifert fibered structures to that of $E(L)$, we see that $E(L) = E(L_1) \cup cl(V - N(L_2))$ is a Seifert fibered space over a disk with two holes with a single singular fiber whose Seifert invariant is $\frac{1}{n}$, and that L is a $(3, 3n)$-torus link. Hence there are infinitely many non-isotopic essential tori in $E(L)$ saturated in the Seifert fibered structure, and those tori are all mutually homeomorphic. Otherwise, any essential torus in $E(L)$ is isotopic to the glueing torus. This completes the proof of Theorem 1.2. □

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