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On composite types of tunnel number two knots

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ABSTRACT

Let K be a tunnel number two knot. Then, by considering the (g, b)-decompositions, K is one of $(3, 0), (2, 1), (1, 2)$ or $(0, 3)$-knots. In this paper, we analyze the connected sum summands of composite tunnel number two knots and give a complete table of those summands from the point of view of (g, b)-decompositions.

Keywords: (g, b)-decompositions; tunnel number two knots; connected sum.

Mathematics Subject Classification 2010: 57M25, 57M27

1. Introduction

Let K be a knot in S^3. Then it is well known that K can be uniquely decomposed into finitely many prime knots, which is due to Schubert [12] and is called the prime decomposition of K. Consider the tunnel number of K denoted by $t(K)$, where the tunnel number is the minimal number of arcs properly embedded in the knot exterior $E(K)$ whose complementary space is homeomorphic to a handlebody. By the definition of the tunnel number, we have $t(K) = g(E(K)) - 1$, where $g(E(K))$ is the Heegaard genus of $E(K)$. For tunnel number one knots, it has been shown by Norwood in [9] that tunnel number one knots are prime. In this paper, we analyze the prime decompositions of tunnel number two knots and give a complete table from the point of view of (g, b)-decompositions defined as below.

Let g and b be non-negative integers with $(g, b) \neq (0, 0)$. Then we say that a knot K has a (g, b)-decomposition if there is a genus g Heegaard splitting (V_1, V_2) of S^3 such that K intersects each handlebody in a b-string trivial arc system. In particular, if $b = 0$ then we define a $(g, 0)$-decomposition as a genus g Heegaard splitting of S^3 such that at least one of the two handlebodies contains K as a central loop of a handle. If $g = 0$, then a $(0, b)$-decomposition is the ordinary b-bridge decomposition.

This decomposition is due to Doll [1] and he showed the additivity of genus g bridge number under the connected sum. This is a generalization of the ordinary bridge decomposition and the additivity due to Schubert [13]. Then, by the definition and a little observation, we see that if a knot K has a (g, b)-decomposition then $t(K) \leq g + b - 1$. This concept, (g, b)-decomposition of knots, plays very important role from the point of view of the tunnel numbers and the distance due to Hempel [2]. For example, see [3], [4] or [5].

Let B be a 3-ball and $t_1 \cup t_2$ be two arcs properly embedded in B. Then $(B, t_1 \cup t_2)$ is called a 2-string tangle. We say that $(B, t_1 \cup t_2)$ is trivial if $t_1 \cup t_2$ is a 2-string trivial arc system in B, that $(B, t_1 \cup t_2)$ is free if $\text{cl}(B - N(t_1 \cup t_2))$ is a genus two handlebody, where $N(t_1 \cup t_2)$ is a regular neighborhood of $t_1 \cup t_2$ in B, that $(B, t_1 \cup t_2)$ is essential if $\text{cl}(\partial B - N(t_1 \cup t_2))$ is incompressible in $\text{cl}(B - N(t_1 \cup t_2))$ and that t_i $(i = 1, 2)$ is unknotted if (B, t_i) is a trivial ball pair. We say that a knot K has a 2-string essential free tangle decomposition if (S^3, K) is decomposed into a union of two 2-string essential free tangles.

For a knot K in S^3, we define the following two conditions $c(1)$ and $c(2)$.

$c(1)$: (S^3, K) has a 2-string essential free tangle decomposition such that exactly one of the two tangles has an unknotted component.

$c(2)$: (S^3, K) has a 2-string essential free tangle decomposition such that each tangle of the two tangles has an unknotted component.

Under the above notations, for composite tunnel number two knots, we have shown the following.

Theorem 1 ([6, 7]). *Let K be a composite tunnel number two knot, then one of the following holds.*

(1) *K is the connected sum of a tunnel number one knot and a knot with a $(1, 1)$-decomposition,*
(2) *K is the connected sum of a 2-bridge knot and a knot with a $c(1)$ or a $c(2)$-condition.*

For composite knots with $(2, 1)$-decompositions, we have shown the following.

Theorem 2 ([8]). *Let K be a composite knot with a $(2, 1)$-decomposition, then one of the following holds.*

(1) *K is the connected sum of a tunnel number one knot and a 2-bridge knot,*
(2) *K is the connected sum of two knots with $(1, 1)$-decompositions,*
(3) *K is the connected sum of a 2-bridge knot and a knot with a $c(2)$-condition.*

For 3-bridge knots, by the additivity of bridge indices due to Schubert, we have the following.

Theorem 3 ([13]). *Let K be a composite 3-bridge knot, then K is the connected sum of two 2-bridge knots.*

In this paper, for knots with $(1,2)$-decompositions, we will show the following.

Theorem 4. *Let K be a composite knot with a $(1,2)$-decomposition, then K is the connected sum of a knot with a $(1,1)$-decomposition and a 2-bridge knot.*

By the way, to state the above results more precisely, we need to define the term (g,b)-knots. To do this, by a little observation, we have the following.

Fact 1. (1) If a knot K has a $(g-1,b+1)$-decomposition, then K has a (g,b)-decomposition.
(2) If a knot K has a $(g,b-1)$-decomposition, then K has a (g,b)-decomposition.

By the above fact, in this paper, we say that a knot K is a (g,b)-knot if K has a (g,b)-decomposition but has neither $(g-1,b+1)$-decomposition nor $(g,b-1)$-decomposition. In addition, we say that a knot K is a $c(i)$-knot if K has a $c(i)$-condition ($i=1,2$). Then the above Theorems 1–4 are rewritten as follows. We note that, since $(3,0)$-knots are tunnel number two knots which have no $(2,1)$-decompositions, it is needed to delete the $c(2)$-condition in Theorem 1. We further note that there is no knot which has both conditions $c(1)$ and $c(2)$ because of the uniqueness of 2-string essential free tangle decompositions due to Ozawa [11].

Theorem 1. *Let K be a composite $(3,0)$-knot, then one of the following holds.*

(1) *K is the connected sum of a $(2,0)$-knot and a $(1,1)$-knot,*
(2) *K is the connected sum of a $(0,2)$-knot and a $c(1)$-knot.*

Theorem 2. *Let K be a composite $(2,1)$-knot, then one of the following holds.*

(1) *K is the connected sum of a $(2,0)$-knot and a $(0,2)$-knot,*
(2) *K is the connected sum of two $(1,1)$-knots,*
(3) *K is the connected sum of a $(0,2)$-knot and a $c(2)$-knot.*

Theorem 3. *Let K be a composite $(0,3)$-knot, then K is the connected sum of two $(0,2)$-knots.*

Theorem 4. *Let K be a composite $(1,2)$-knot, then K is the connected sum of a $(1,1)$-knot and a $(0,2)$-knot.*

Then, by summarizing the above results, we have Table 1 of composite tunnel number two knots from the point of view of (g,b)-decompositions.

Table 1. Composite types of tunnel number two knots.

(g,b)	Composite types
$(3,0)$	$(2,0)\#(1,1)$ or $(0,2)\#c(1)$
$(2,1)$	$(2,0)\#(0,2)$, $(1,1)\#(1,1)$ or $(0,2)\#c(2)$
$(1,2)$	$(1,1)\#(0,2)$
$(0,3)$	$(0,2)\#(0,2)$

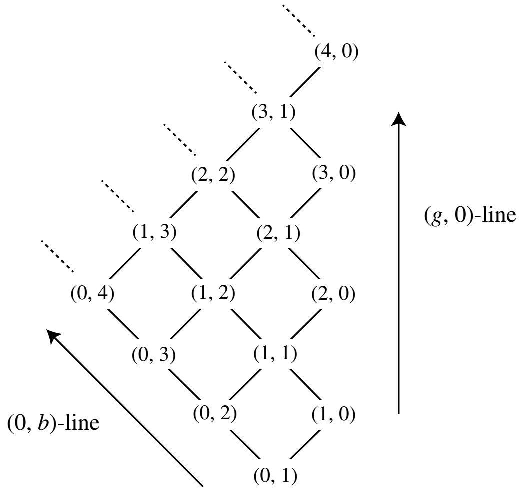

Fig. 1. The (g, b)-diagram.

Remark 1. Concerning the $c(1)$ and $c(2)$-conditions, we have the following:

(1) If a knot K has $c(1)$-condition then K is a prime $(3, 0)$-knot. Hence $(0, 2)\#c(1)$ is included in $(0, 2)\#(3, 0)$ and this is the tunnel number degeneration "$2+1 = 2$" in [7].

(2) If a knot K has $c(2)$-condition then K is a prime $(2, 1)$-knot. Hence $(0, 2)\#c(2)$ is included in $(0, 2)\#(2, 1)$ and this is also the tunnel number degeneration "$2 + 1 = 2$" in [7].

Remark 2. By Fact 1, the family of knots with (g, b)-decompositions contains the family of knots with $(g - 1, b + 1)$-decompositions and the family of knots with $(g, b-1)$-decompositions. Hence we have the (g, b)-diagram as illustrated in Fig. 1.

2. Proof of Theorem 4

Let K be a composite knot in S^3 with the decomposing 2-sphere S. Suppose K has a $(1, 2)$-decomposition. Then there is a genus one Heegaard splitting (V_1, V_2) of S^3 such that K intersects V_i $(i = 1, 2)$ in 2-string trivial arc system, where V_i is a solid torus.

Put $V_i \cap K = \gamma_i^1 \cup \gamma_i^2$, and $S_i = V_i \cap S$. Then, by taking a spine of V_1, we may assume that S_1 consists of disks not intersecting $\gamma_1^1 \cup \gamma_1^2$ and S_2 is a planar surface properly embedded in V_2 intersecting $\gamma_2^1 \cup \gamma_2^2$ in two points. In addition we may assume that the number of the components of S_1 is minimal among all decomposing 2-spheres of K. Then we have the following.

Lemma 2.1. *Each component of S_1 is one of the following three types as in Fig.* 2:

(1) *a separating disk which cuts off a* 3*-ball containing one of $\gamma_1^1 \cup \gamma_1^2$*,

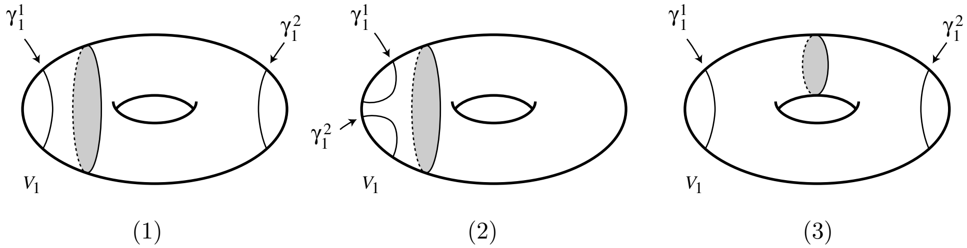

Fig. 2. Disks in V_1.

(2) *a separating disk which cuts off a* 3*-ball containing both of* $\gamma_1^1 \cup \gamma_1^2$,
(3) *a non-separating disk.*

Proof. Let D be a component of S_1. Suppose D is a separating disk. Then D divides V_1 into a 3-ball and a solid torus. If the 3-ball contains no component of $\gamma_1^1 \cup \gamma_1^2$, then we can reduce the number of the components of S_1 and this contradicts the minimality. Thus the 3-ball contains at least one component of $\gamma_1^1 \cup \gamma_1^2$ and this completes the proof of the lemma. □

Next, let E_1 and E_2 be the disks in V_2 for the triviality of γ_2^1 and γ_2^2 respectively, and E_3 and E_4 the two non-separating disks in V_2 such that $E_3 \cup E_4$ divides V_2 into two 3-balls each of which contains one of $E_1 \cup E_2$ as in Fig. 3.

Put $E = E_1 \cup E_2 \cup E_3 \cup E_4$, then by standard cut and paste operations, we may assume that each component of $S_2 \cap E$ is an arc properly embedded in E. We say that an arc α properly embedded in S_2 is γ-essential if α is essential in $S_2 - (\gamma_2^1 \cup \gamma_2^2)$. Suppose there is an arc α in $S_2 \cap E$ which is γ-essential in S_2. Let Δ be the disk in E such that $\partial\Delta$ is the union of α and a subarc of $\partial E - (\gamma_2^1 \cup \gamma_2^2)$. We may assume that $\Delta \cap S_2 = \alpha$ by changing the disks E if necessary. Then we can perform a boundary compression of S_2 at α along Δ from V_2 to V_1, and we get a band, say b, in V_1. If b connects two different disks in S_1, then we can reduce the number of the components of S_1 and this contradicts the minimality. Thus b

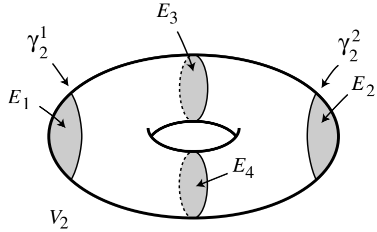

Fig. 3. $E_1 \cup E_2 \cup E_3 \cup E_4$ in V_2.

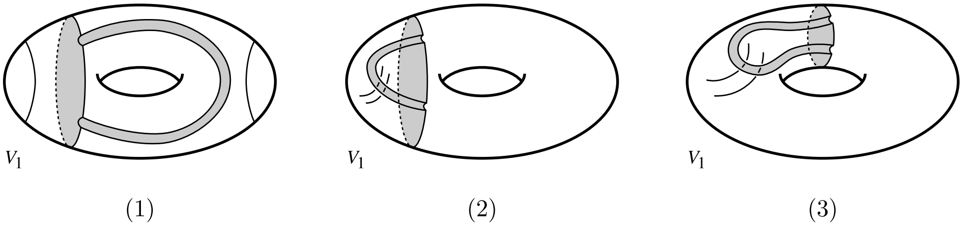

Fig. 4. Annuli in V_1.

connects a single disk, and the union of the band and the disk is an annulus in V_1. Then we have the following.

Lemma 2.2. *The annulus is one of the following three types as in Fig.* 4:

(1) *the union of a separating disk of type* (1) *in Lemma* 2.1 *and a band which is contained in the solid torus component and winds around the longitude exactly once*,
(2) *the union of a separating disk of type* (2) *in Lemma* 2.1 *and a band which is contained in the* 3*-ball component such that the compressing disk of the annulus in* ∂V_1 *intersects* $\gamma_1^1 \cup \gamma_1^2$ *in two points.*
(3) *the union of a non-separating disk and a band such that the compressing disk of the annulus in* ∂V_1 *intersects* $\gamma_1^1 \cup \gamma_1^2$ *in two points*,

Proof. Let D be a disk component of S_1 which is connected to the band b, then $D \cup b$, say A, is an annulus properly embedded in V_1. Suppose D is a separating disk of type (1). Then, since D divides V_1 into a 3-ball and a solid torus, b is contained in the 3-ball or in the solid torus. If b is contained in the 3-ball, then A is compressible in V_1 and by standard cut and paste operation, we can take another decomposing 2-sphere which intersects V_1 in fewer disk components. This contradicts the minimality. Thus b is contained in the solid torus component. Then by the same reason as above, b winds around the solid torus at least once in the longitudinal direction. However, if the band winds more than once, then, since we can regard each disk of $S - A$ is a 2-handle for V_1, we have the lens space not S^3. This is a contradiction. Thus b winds around V_1 exactly once and A is an annulus of type (1).

Next suppose D is a separating disk of type (2). Then, since D divides V_1 into a 3-ball and a solid torus, b is contained in the 3-ball or in the solid torus. If b is contained in the solid torus, then by the same reason as above b winds around the solid torus exactly once. However, in this case, A can be pushed out into V_2 and we can reduce the number of the components of S_1. Thus b is contained in the 3-ball component. Then each component of ∂A bounds a disk in ∂V_1. If at least one disk intersects $\gamma_1^1 \cup \gamma_1^2$ in 0 or 1 point, then we can take another decomposing 2-sphere which intersects V_1 in fewer disk components. This contradicts the minimality and shows that A is an annulus of type (2).

Finally suppose D is a non-separating disk. Then, since D cuts open V_1 into a 3-ball, A is compressible in V_1. Let Δ be a compressing disk for A in ∂V_1. If $\Delta \cap (\gamma_1^1 \cup \gamma_1^2) = 0$ or 1 point, then by cut and paste operation we can take another decomposing 2-sphere which intersects V_1 in fewer disk components. This contradicts the minimality. Thus $\Delta \cap (\gamma_1^1 \cup \gamma_1^2) = 2$, 3 or 4 points. However, if it is 3 or 4 points, then by taking another compressing disk for A, i.e. a meridian disk of V_1, we can reduce the number of the components of S_1 and have a contradiction similarly. Thus A is an annulus of type (3), and this completes the proof. □

Let n be the number of the components of S_1, then we have the following.

Lemma 2.3. *We have $n = 1$. Hence S_1 is a single disk not intersecting $\gamma_1^1 \cup \gamma_1^2$ and S_2 is a single disk intersecting $\gamma_2^1 \cup \gamma_2^2$ in two points.*

Proof. Suppose $n > 1$. Then, since S_2 has a non-trivial fundamental group, $S_2 \cap E$ has a γ-essential arc. Thus we can perform a boundary compression and get a band b_1 in V_1. Then by the minimality of S_1, b_1 connects a single component of S_1 and an annulus is produced in V_1. Continue this procedure. Then, at some kth stage, we have that $b_1, b_2, \ldots, b_k$ are the bands each of which connects a single component of S_1 and b_{k+1} connects two different components of S_1, where to avoid the confusion of notations we use the same notations of S_1 and S_2 even after the boundary compressions. We note that there are no two bands b_i and b_j which connect the same disk because at each stage core arcs of those bands are γ-essential in S_2. Then, at this stage, we have k annuli $A_1, A_2, \ldots, A_k$ in V_1.

Suppose $k < n$. Then, since there remains a disk component of S_1, we have the following two cases:

(i) b_{k+1} connects a disk and an annulus A_i,
(ii) b_{k+1} connects two annuli A_i, A_j $(i < j)$.

In case (i), we can use the inverse operation of boundary compression introduced by Ochiai in [10], and can reduce the number of the components of S_1. This is a contradiction. In case (ii), we have two subcases, (ii-a) $A_1, A_2, \ldots, A_k$ are all mutually parallel annuli of type (1) in Lemma 2.2, (ii-b) $A_1, A_2, \ldots, A_k$ are of type (2) or of type (3) in Lemma 2.2. Then, in case (ii-a), b_{k+1} does not run over the band b_j. In case (ii-b), by the existence of the disk of type (3) in Lemma 2.1, we see that b_{k+1} does not run over the band b_j. Then, in both cases, we can pull back the bands $b_j, \ldots, b_k$ leaving b_{k+1} in V_1. This means that case (ii) is reduced to case (i) and we have a contradiction. Thus we have $k = n$.

By the above arguments, we can put $S_1 = A_1 \cup A_2 \cup \cdots \cup A_n$ and $S_2 = D_1^* \cup D_2^* \cup B_1 \cup B_2 \cup \cdots \cup B_{n-1}$, where A_i $(i = 1, 2, \cdots, n)$ is an annulus in V_1, B_i

$(i = 1, 2, \cdots, n-1)$ is an annulus in V_2 and D_i^* $(i = 1, 2)$ is a disk in V_2 intersecting $\gamma_2^1 \cup \gamma_2^2$ in a single point.

Suppose D_1^* is a non-separating disk and ∂D_1^* is identified with a component of ∂A_i for some i. Then, since $V_1 \cup V_2 = S^3$, A_i is of type (1) in Lemma 2.2. Then, by Lemma 2.2, those components of S_1 are all mutually parallel annuli of type (1). This means that each component of ∂S_2 is a meridian of V_2. However, each annulus component of S_2 has a boundary component which is not a meridian by Lemma 2.2. This is a contradiction, and shows that both of D_1^* and D_2^* are separating disks.

Let X be a 3-ball in V_2 cut off by D_1^*. Then we may assume that $X \cap S_2 = D_1^*$, and since $D_i^* \cap (\gamma_2^1 \cup \gamma_2^2)$ $(i = 1, 2)$ is a single point, $(X \cap \partial V_2) \cap (\gamma_2^1 \cup \gamma_2^2)$ is a single point or three points. However, there is no annulus component of S_1 whose boundary component bounds a disk in ∂V_1 intersecting $\gamma_1^1 \cup \gamma_1^2$ in a single point or in three points. This contradiction is due to the hypothesis $n > 1$, and completes the proof. □

Proof of Theorem 4. By Lemma 2.3, S_1 is a single separating disk not intersecting $\gamma_1^1 \cup \gamma_1^2$, and S_2 is a single separating disk intersecting $\gamma_2^1 \cup \gamma_2^2$ in two points. We may assume that $S_2 \cap E$ consists of arcs properly embedded in E and that the number of the components of $S_2 \cap E$ is minimal among all decomposing 2-spheres intersecting V_i $(i = 1, 2)$ in a single disk. We note that $S_2 \cap E$ contains two arcs which meets a point of $S_2 \cap (\gamma_2^1 \cup \gamma_2^2)$. Then we have the following two cases:

(i) $S_2 \cap E$ consists of exactly two arcs each of which meets a point of $S_2 \cap (\gamma_2^1 \cup \gamma_2^2)$,
(ii) $S_2 \cap E$ contains a γ-essential arc properly embedded in S_2.

Suppose we are in case (i). Then we have an arc component α of $S_2 \cap E_1$ which intersects γ_2^1 in a single point. Let Δ be one of the two disks in E_1 cut off by α. Then we may assume $\Delta \cap S = \alpha$, and can isotope S along the disk Δ so that $S \cap V_i$ $(i = 1, 2)$ is a single separating disk intersecting $\gamma_i^1 \cup \gamma_i^2$ in a single point as in Fig. 5.

Let X_i $(i = 1, 2)$ be the 3-ball in V_i cut off by S_i. Since, $S_i \cap (\gamma_i^1 \cup \gamma_i^2)$ is a single point, $(X_i \cap \partial V_i) \cap (\gamma_i^1 \cup \gamma_i^2)$ is one or three points. If it is one point, then the tangle $(X_i, X_i \cap (\gamma_i^1 \cup \gamma_i^2))$ is a trivial tangle. In this case, $(X_1 \cup X_2, (X_1 \cup X_2) \cap K)$ is a 1-string trivial tangle and this means that S bounds a trivial ball pair. This is a contradiction because S is a decomposing 2-sphere of the non-trivial connected sum of K. Thus we have that $(X_i \cap \partial V_i) \cap (\gamma_i^1 \cup \gamma_i^2)$ consists of three points as in Fig. 6.

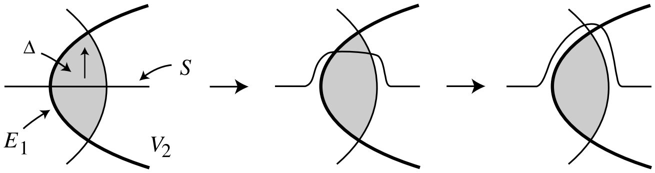

Fig. 5. The isotopy along Δ.

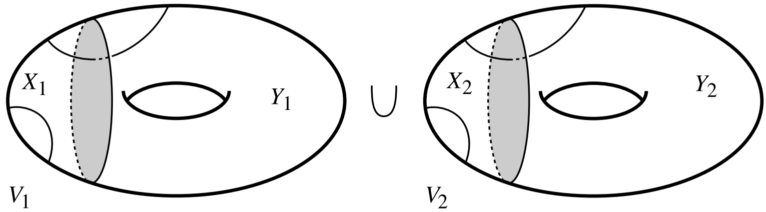

Fig. 6. The intersection of S and $V_1 \cup V_2$ in case (i).

Let X_i $(i = 1, 2)$ be the 3-ball in V_i cut off by S_i and Y_i the solid torus in V_i cut off by S_i. Then $(X_i, X_i \cap (\gamma_i^1 \cup \gamma_i^2))$ is a 2-string trivial tangle and $(Y_i, Y_i \cap (\gamma_i^1 \cup \gamma_i^2))$ is a solid torus with a single trivial arc. Hence $X_1 \cup X_2$ extends to a 2-bridge decomposition of a knot and $Y_1 \cup Y_2$ extends to a knot with a (1, 1)-decomposition, i.e. K is the connected sum of a (0, 2)-knot and a (1, 1)-knot.

Next, suppose we are in case (ii). Let α be a γ-essential arc properly embedded in S_2. Then we may assume that α is outermost in E. Perform a boundary compression of S_2 at α. Then, since α is an arc in S_2 which splits the two points $S_2 \cap (\gamma_2^1 \cup \gamma_2^2)$, we can put $S_1 = A$ and $S_2 = D_1^* \cup D_2^*$, where A is an annulus of type (1) in Lemma 2.2 and D_i^* $(i = 1, 2)$ is a meridian disk of V_2 intersecting $\gamma_2^1 \cup \gamma_2^2$ in a single point. We note that if A is an annulus of type (2) in Lemma 2.2, then D_i^* is a separating disk intersecting $\gamma_i^1 \cup \gamma_i^2$ in two points and this is a contradiction.

Let $X_i \cup Y_i$ $(i = 1, 2)$ be the two components cut off by S_i, where X_1 and X_2 are identified and Y_1 and Y_2 are identified as in Fig. 7. Then, by adding a 2-handle to X_1 along the annulus A, (X_1, γ_1^1) extends to a 2-string trivial tangle, where a core arc of the 2-handle is regarded as a string. In addition $(X_2, X_2 \cap (\gamma_2^1 \cup \gamma_2^2))$ is also a 2-string trivial tangle.

Hence $X_1 \cup X_2$ extends to a 2-bridge decomposition of a knot. On the other hand, (Y_1, γ_1^2) is a solid torus with a trivial arc, and by adding a 1-handle to Y_2 along the two disks $D_1^* \cup D_2^*$, $(Y_2, Y_2 \cap (\gamma_2^1 \cup \gamma_2^2))$ extends to a solid torus with a trivial arc, where a core arc of the 1-handle is regarded as a part of the trivial arc. Hence $Y_1 \cup Y_2$ extends to a $(1, 1)$-decomposition of a knot. This shows that K is the connected sum of a (0, 2)-knot and a (1, 1)-knot, and completes the proof of Theorem 4. □

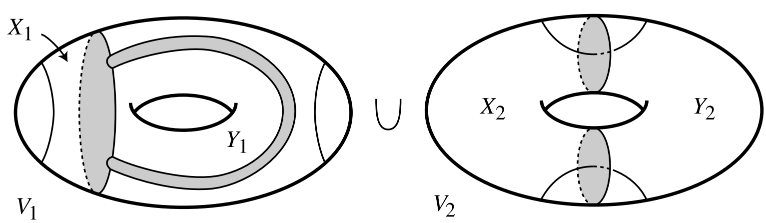

Fig. 7. The intersection of S and $V_1 \cup V_2$ in case (ii).

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