On 2-component Links with Genus Two Heegaard Splittings

by

Kanji Morimoto

Department of IS and Mathematics, Konan University
Okamoto 8-9-1, Higashi-Nada, Kobe 658-8501, Japan
morimoto@konan-u.ac.jp

Abstract

In the present paper, we consider two types of 2-component links with genus two Heegaard splittings. One of them is an ordinary tunnel number one link, and the other is a somewhat different tunnel nuber one link. We will try to detect the differences between those two types. In fact, we will characterize composite tunnel number one links of the second type, and tunnel number one links of the second type with essential tori.

Keywords: genus two Heegaard splittings, tunnel number one links, connected sum, composite, essential tori, essential tangle decompositions

2010 Mathematics Subject Classification : 57M25, 57M27

1. Introduction

Let $L = K_1 \cup K_2$ be a 2-component link in S^3. Let $N(L) = N(K_1) \cup N(K_2)$ be a regular neighborhood of L in S^3 and $E(L) = cl(S^3 - N(L))$ the exterior. Then $\partial E(L) = \partial N(K_1) \cup \partial N(K_2)$ is two tori. Suppose $E(L)$ has a genus two Heegaard splitting $E(L) = C_1 \cup C_2$, where $C_i (i = 1, 2)$ is a genus two handlebody or a genus two compression body. Then we have the following two cases : one of them is that $\partial E(L)$ is contained in ∂C_1 or in ∂C_2, and the other is that, by changing the letters if necessary, $\partial N(K_1)$ ($\partial N(K_2)$ resp.) is contained in ∂C_1 (∂C_2 resp.).

So far, we say that L has tunnel number one if the former case occurs, and it seems that the latter case has not been much studied. See [2, 9, 11] for example. In knot case, there are no such ambiguities, but in link case, we need to consider the differences between these two cases. In the present paper, we study these two cases of those links and try to detect the differences. So we will define the two types of tunnel number one links as follows.

We say that L is a tunnel number one link of type I if there is an arc γ in S^3 such that $L \cap \gamma = \partial\gamma$, γ connects K_1 and K_2 and the exterior $E(K_1 \cup \gamma \cup K_2)$ is a genus two handlebody (Figre 1-(1)), and that L is a tunnel number one link of type II if there is an arc γ in S^3 such that, by changing the letters if necessary, $L \cap \gamma = K_1 \cap \gamma = \partial\gamma$

and the exterior $E(K_1 \cup \gamma)$ is a genus two handlebody containing K_2 as a core of a handle (Figre 1-(2)).

In case of type I we call γ an unknotting tunnel of type I, and in case of type II we call γ an unknotting tunnel of type II. Put $V_1 = N(K_1 \cup \gamma \cup K_2)$ and $V_2 = cl(S^3 - V_1)$ in case of type I, and put $V_1 = N(K_1 \cup \gamma)$ and $V_2 = cl(S^3 - V_1)$ in case of type II. Then in both cases, (V_1, V_2) is a genus two Heegaard splitting of S^3 as in Figure 1.

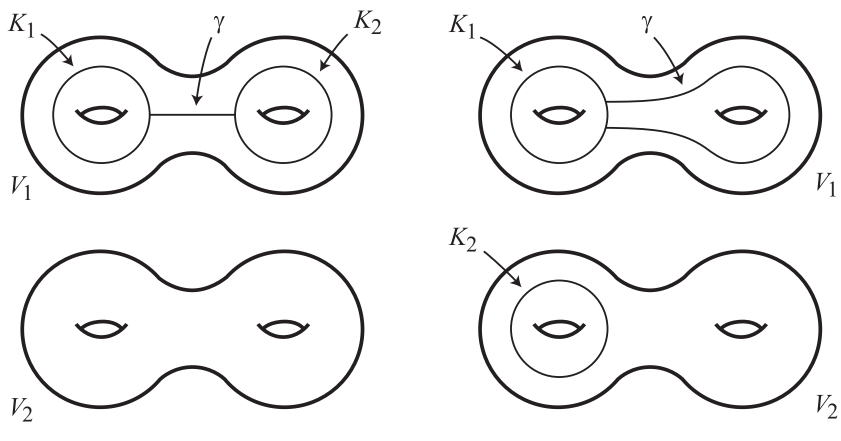


Figure 1 : Heegaard splittings and unknotting tunnels

On inclusion relations of type I and type II, Ishihara showed in [5] the following :

Theorem 1 (A part of Theorems 1.4 and 1.5 of [5]) (1) *There are infinitely many tunnel number one links of type I not of type II.* (2) *There are infinitely many tunnel number one links of type II not of type I.*

By this theorem, we see that two families of tunnel number one links of type I and of type II are independent. Of course the intersection of these two families is not empty, for example, 2-bridge links are tunnel number one links of both types. In the following we characterize tunnel number one links of type II with some conditions.

First, we consider composite tunnel number one links. We say that a knot K is a 2-bridge knot if there is a genus zero Heegaard splitting (B_1, B_2) of S^3 such that $K \cap B_i$ is a 2-string trivial arc system properly embedded in $B_i (i = 1, 2)$, and that a knot K is a (1, 1)-knot if there is a genus one Heegaard splitting (V_1, V_2) of S^3 such that $K \cap V_i$ is a trivial arc properly embedded in $V_i (i = 1, 2)$. In [9], we characterized composite tunnel number one links of type I as follows :

Theorem 2 (Theorem 1 of [9]) *Let L be a tunnel number one link of type I. Then L is composite if and only if L is a connected sum of a $2-$bridge knot and a Hopf link.*

In section 2, we will characterize composite tunnel number one links of type II as follows :

Theorem 3 *Let L be a tunnel number one link of type II. Then L is composite if and only if L is a connected sum of a $(1,1)-$knot and a Hopf link.*

By these two theorems and the fact that 2-bridge knots are (1, 1)-knots, we see that the family of composite tunnel number one links of type I is properly contained in the family of composite tunnel number one links of type II, and that the difference is corresponding to the difference between 2-bridge knots and (1, 1)-knots.

Next, we consider tunnel number one links with essential tori. Let $L = K_1 \cup K_2$ be a 2-bridge link not a Hopf link or a trivial link. Since K_2 is a trivial knot, $E(K_2)$ is a solid torus with $K_1 \subset E(K_2)$. Let $T(p,q)$ be a torus knot of type (p,q) for some relatively prime integers p,q with $|p| > 1$ and $|q| > 1$, and let $N(p,q)$ be a regular neighborhood of $T(p,q)$ and $E(p,q) = cl(S^3 - N(p,q))$ the exterior. Then $E(p,q)$ is a Seifert fibered space over a disk with two exceptional points $D(-r/p, s/q)$ with $ps + qr = 1$. Let m be a longitude of the solid torus $E(K_2)$ in $\partial E(K_2)$ which is a merdian of K_2, and ℓ a regular fiber of $D(-r/p, s/q)$ in $\partial E(p,q)$, then we have an orientation preserving homeomorphism $f : E(K_2) \to N(p,q)$ with $f(m) = \ell$. Then we have a knot $f(K_1) \subset f(E(K_2)) = N(p,q) \subset S^3$ and call $f(K_1)$ an MS-knot. Then since L is not a Hopf link or a trivial link and since $T(p,q)$ is a non-trivial torus knot, MS-knot contains an essential torus in the exterior. Then in [10] we showed the following :

Theorem 4 (Theorem A of [10]) *Let K be a tunnel number one knot. Then K has an essential torus in the exterior if and only if K is an MS-knot.*

As a link version of the above theorem, Munoz and Uchida showed the follwing :

Theorem 5 (Theorems 1.2 of [11] *Let L be a tunnel number one link of type I. Then L has an essential torus in the exterior if and only if one of the following holds:*

$C(1)$: L is a connected sum of a $2-$bridge knot and a Hopf link,

$C(2)$: L is an union of an MS-knot and a exceptional fiber of the Seifert fibered space $D(-r/p, s/q)$.

In section 3, we will characterize tunnel number one links of type II with essential tori. Let's consider a 3-bridge link $L = K_1 \cup K_2 \cup K_3$ with the follwing conditions :

(i) $K_1 \cup K_2$ is a Hopf link,

(ii) both $K_1 \cup K_3$ and $K_2 \cup K_3$ are non-trivial 2-bridge links.

Then, since K_3 is a trivial knot, $E(K_3)$ is a solid torus with $K_1 \cup K_2 \subset E(K_3)$. Let m be a longitude of the solid torus $E(K_3)$ in $\partial E(K_3)$ which is a merdian of K_3, and let $T(p,q)$, $N(p,q)$, $E(p,q)$ and ℓ be as above. Then we have an orientation preserving homeomorphism $f : E(K_3) \to N(p,q)$ with $f(m) = \ell$. Then we have a link $f(K_1 \cup K_2) \subset f(E(L_3)) = N(p,q) \subset S^3$ and call $f(K_1 \cup K_2)$ an MS-link. Then by the above conditions and that $T(p,q)$ is a non-trivial knot, MS-link contains an essential torus in the exterior, and is not a composite link. The link illustrated in Figure 2 is an example of an MS-link.

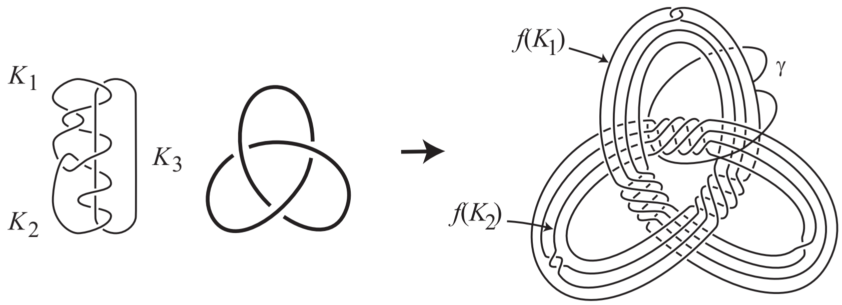


Figure 2 : MS-link

Then we show the following :

Theorem 6 *Let L be a tunnel number one link of type II. Then L has an essential torus in the exterior if and only if one of the following holds :*

$C(1)$: *L is a connected sum of a $(1,1)-$knot and a Hopf link,*

$C(2)$: *L is an union of an MS-knot and an exceptional fiber of the Seifert fibered space $D(-r/p, s/q)$,*

$C(3)$: *L is an MS-link.*

By these two theorems, we see that the family of tunnel number one links of type I with essential tori is properly contained in the family of tunnel number one links of type II with essential tori. In particular, $C(3)$ is disjoint from $C(1) \cup C(2)$.

Finally, we consider tunnel number one links with essential tangle decompositions. We say that a link L has an n-string essential tangle decomposition for $n > 0$ if there is a genus zero Heegaard spliting (B_1, B_2) of S^3 such that $(B_i, L \cap B_i)$ is an essential tangle for both $i = 1, 2$, where $(B_i, L \cap B_i)$ is essential if $\partial B_i - L$ is incompressible in $B_i - L$ for $n > 1$ and $L \cap B_i$ is not a trivial arc in B_i for $n = 1$. Then Gordon and Reid showed in [2] the following :

Theorem 7 (A part of Theorem 1.5 of [2]) *If a tunnel number one link L of type I has an essential tangle decomposition, then at least one of the two components of L is a trivial knot.*

On tangle decompositions of tunnel number one links of type II, Ishihara showed in

[5] that 2-component Montesinos link $M(b; \frac{a_1}{b_1}, \frac{1}{2}, \frac{a_2}{b_2}, \frac{1}{2})$ is a tunnel number one link of type II not of type I, where (a_i, b_i) is a pair of relatively prime integers for $i = 1, 2$. Then we see that this link has a 2-string essential tangle decomposition, and that both components of this link are non-trivial 2-bridge knots.

In section 4, we will show that there are infinitely many tunnel number one links of type II not of type I with n-string essential tangle decomposition for any $n > 0$, each of which consists of a non-trivial 2-bridge knot and a non-trivial (1, 1)-knot.

We note that Ishihara showed in [5] that if a tunnel number one link of type I has a trivial component then it is of type II too (Theorem 1.7 of [5]). Thus by combining these theorems and examples, we see that the family of tunnel number one links of type I with essential tangle decompositions is properly contained in the family of tunnel number one links of type II with essential tangle decompositions.

So far, we have considered tunnel number one links with three conditions : composite, with essential tori and with essential tangle decompositions, and have seen the differences between tunnel number one links of type I and of type II. Although Theorem 1 says that there are infinitely many tunnel number one links of type I not of type II, we cannot get concrete examples of such links yet. So we end Introduction by putting the following problems :

Problems

(1) Construct concrete examples of tunnel number one links of type I not of type II.

(2) Show the type II version of Gordon-Reid's theorem.

In the present paper, for standard terms and definitions in knot theory and 3-manifold topology, we refer to [4, 6, 12].

2. Proof of Theorem 3

Let $L = K_1 \cup K_2$ be a composite tunnel number one link of type II, and let S be a decomposing 2-sphere with $S \cap K_1 =$ two points and $S \cap K_2 = \emptyset$. Let γ be an unknotting tunnel of type II of L with $K_1 \cap \gamma = \partial\gamma$. Put $V_1 = N(K_1 \cup \gamma)$ and $V_2 = cl(S^3 - V_1)$, then (V_1, V_2) is a genus two Heegaard splitting of S^3 as illustrated in Figure 1-(2).

By $S \cap K_1 =$ two points, we may assume that $S \cap V_1 = D_1^* \cup D_2^* \cup D_1 \cup D_2 \cup \cdots \cup D_n$, where $D_i^* (i = 1, 2)$ is a meridian disk with $D_i^* \cap K_1 =$ one point and $D_j (j = 1, 2, \cdots n)$ is a disk properly embedded in V_1 not ∂-parallel as illustrated in Figure 3.

Put $P = S \cap V_2$, then P is a planar surface properly embedded in V_2 with $n + 2$ boundary components. Suppose n is minimal among all such decomposing 2-spheres. Then we may assume that P is incompressible in V_2. Let E be a meridian disk of V_2 with $E \cap K_2 = \emptyset$, and suppose $n > 0$. Then, since ∂P consists of $n + 2 (> 2)$

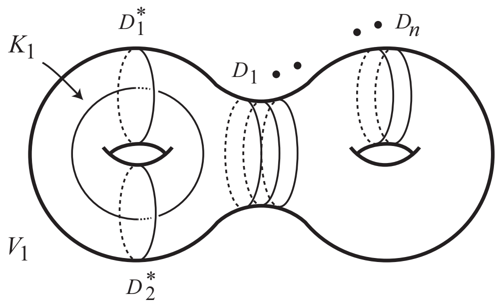


Figure 3 : $S \cap V_1$

components, $P \cap E \neq \emptyset$ and we may assume that each component of $P \cap E$ is an arc properly embedded in both P and E.

Let α be an outermost arc component of $P \cap E$ in E, and let Δ be the corresponding outer most disk. Then we can perform a boundary compression of P at α along Δ from V_2 to V_1, and we get a band, say b, in V_1, If b connects two different disks, then we can reduce the number of the components of $S \cap V_1$, and this contradicts the minimality of n. Thus b connects a single disk. If there is a non-separating disk in $D_1 \cup D_2 \cup \cdots \cup D_n$, then b connects D_1 and $b \cup D_1$ is a compressible annulus in V_1 which has a compressing disk intersecting K_1 in a single point. Then by preforming a surgery along the disk, we get a decomposing 2-sphere intersecting V_1 in fewer essential disks than n. This contradicts the minimality of n. Thus $D_1 \cup D_2 \cup \cdots \cup D_n$ are all mutually parallel separating disks. Then we may assume that b connects the separating disk D_n as in Figure 4.

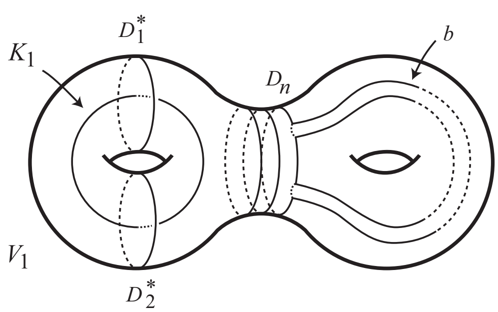
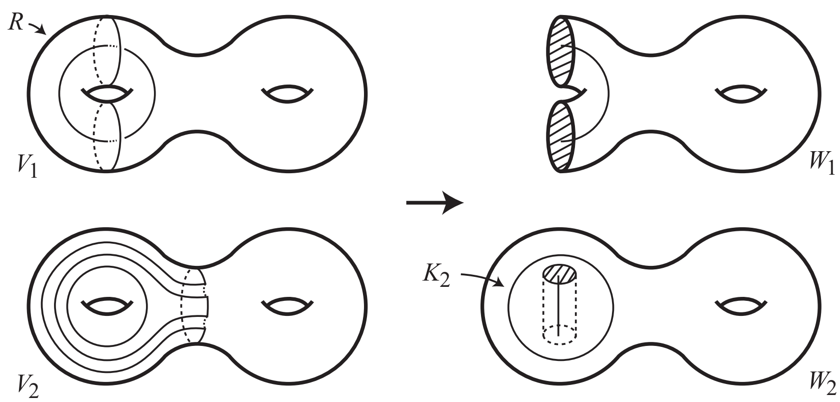


Figure 4 : $D_n \cup b$

Suppose b winds around a handle of V_1 p times for some $p > 0$. Then, since $cl(S - (D_n \cup b))$ consists of two disks, the union of the solid torus cut off by $D_n \cup b$ and one of the two disks shows that S^3 contains a lens space summand of the order p. Hence $p = 1$, and the annulus $D_n \cup b$ is a ∂-parallel annulus. Then we can reduce the number of the components of $S \cap V_1$, and this contradicts the minimality of n.

After all, we have $n=0$ and $S\cap V_1 = D_1^*\cup D_2^*$. Thus $S\cap V_2$ is a separating annulus consisting of a separating disk and a band winding around a handle containing K_2 exactly once as in Figure 5-(1).



(1) (2)

Figure 5 : $(V_1\cup V_2)\to(W_1\cup W_2)$

Let R be the 3-ball in V_1 cut off by $D_1^*\cup D_2^*$ indicated in Figure 5-(1). Put $W_1 = cl(V_1-R)$ and $W_2=V_2\cup R$ as in Figure 5-(2). Then W_1 is a solid torus, and since the annulus $S\cap V_2$ winds around a handle containing K_2 once, W_2 is a solid torus too. Then, since $W_1\cap K_1$ is a trivial arc in W_1 and $W_2\cap K_1$ is a trivial arc in W_2, K_1 is a (1, 1)-knot. Moreover, K_2 is a trivial loop in W_2 bounding a disk intersecting K_1 in a single point. This shows that L is a connected sum of a (1, 1)-knot and a Hopf link.

On the other hand, the converse is proved by tracing back the above arguments, and this completes the proof of Theorem 3. □

3. Proof of Theorem 6

Befor the proof of Theorem 6, we prepare the following lemma :

Lemma 8 *Let $L=K_1\cup K_2\cup K_3$ be a 3-component link with the following conditions:*
(i) there is a genus one Heegaard splitting (V_1,V_2) of S^3 such that K_i is a core of $V_i (i=1,2)$,
(ii) K_3 is a trivial knot,
(iii) K_3 intersects V_i in a trivial arc and there is a trivializing disk Δ_i for $K_3\cap V_i$ with $\Delta_i\cap K_i=\emptyset (i=1,2)$ as in Figure 6.
Then $K_1\cup K_2$ is a Hopf link and L is a 3-bridge link.

Proof. By the condition (i), it is clear that $K_1\cup K_2$ is a Hopf link.

Let D_1 be a meridian disk of V_1 such that D_1 intersects K_1 in a single point

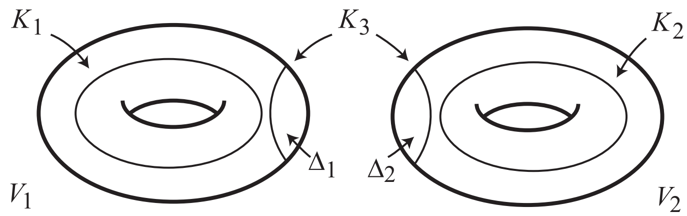


Figure 6 : $K_1 \cup K_2 \cup K_3$

and $D_1 \cap \Delta_1 = \emptyset$. Let $N(D_1)$ be a regular neighborhood of D_1 and put $U_1 = cl(V_1 - N(D_1)), U_2 = V_2 \cup N(D_1)$. Then both U_1 and U_2 are 3-balls and (U_1, U_2) is a genus zero Heegaard splitting of S^3. Put $K_1 \cap U_1 = \alpha_1$, $K_1 \cap U_2 = \alpha_2$, $K_3 \cap U_1 = \gamma_1$ and $K_3 \cap U_2 = \gamma_2$. Then by the construction $(U_1, \alpha_1 \cup \gamma_1)$ is a 2-string trivial arc system in U_1. On the other hand, $\pi_1(U_2 - (\alpha_2 \cup \gamma_2)) \cong Z * Z$, $\pi_1(U_2 - \alpha_2) \cong Z$, and we have $\pi_1(U_2 - \gamma_2) \cong Z$ because K_3 is a trivial knot. Then by [1, Theorem 1], $(U_2, \alpha_2 \cup \gamma_2)$ is a 2-string trivial arc system in U_2. This shows that $K_1 \cup K_3$ is a 2-bridge link. By the same reason, $K_2 \cup K_3$ is a 2-bridge link too.

Let D_2 be a meridian disk of V_2 such that D_2 intersects K_2 in a single point and $D_2 \cap \Delta_2 = \emptyset$. Let $N(D_2)$ be a regular neighborhood of D_2 in V_2. Put $W_1 = U_1 \cup N(D_2), W_2 = cl(U_2 - N(D_2))$. Then both W_1 and W_2 are 3-balls and (W_1, W_2) is a genus zero Heegaard splitting of S^3. Put $K_2 \cap W_1 = \beta_1$ and $K_2 \cap W_2 = \beta_2$. Then by the construction, $(W_1, \alpha_1 \cup \beta_1 \cup \gamma_1)$ is a 3-string trivial arc system in W_1. On the other hand $\pi_1(W_2 - (\alpha_2 \cup \beta_2 \cup \gamma_2)) \cong Z * Z * Z$, $\pi_1(W_2 - (\alpha_2 \cup \beta_2)) \cong Z * Z$, $\pi_1(W_2 - (\alpha_2 \cup \gamma_2)) \cong Z * Z$, and we have $\pi_1(W_2 - (\beta_2 \cup \gamma_2)) \cong Z * Z$ because $K_2 \cup K_3$ is a 2-bridge link as we showed above and $(W_2, \beta_2 \cup \gamma_2)$ corresponds to a trivial tangle of the 2-bridge decompositin. Moreover, $\pi_1(W_2 - \alpha_2) \cong Z$, $\pi_1(W_2 - \beta_2) \cong Z$ and $\pi_1(W_2 - \gamma_2) \cong Z$ because K_1, K_2, K_3 are all trivial knots. Then by [1, Theorem 1], $(W_2, \alpha_2 \cup \beta_2 \cup \gamma_2)$ is a 3-string trivial arc system in W_2. This shows that $L = K_1 \cup K_2 \cup K_3$ is a 3-bridge link and completes the proof of Lemma 8. □

Now, we prove Theorem 6. Let $L = K_1 \cup K_2$ be a tunnel number one link of type II with an essential torus, and let T be the essential torus in the exterior. Let γ be an unknotting tunnel of type II of L with $K_1 \cap \gamma = \partial\gamma$, and put $V_1 = N(K_1 \cup \gamma)$ and $V_2 = cl(S^3 - V_1)$. Then (V_1, V_2) is a genus two Heegaard splitting of S^3 as illustrated in Figure 1-(2).

By considering the intersections of T and $K_1 \cup \gamma$, we may assume that $T \cap V_1 = D_1 \cup D_2 \cup \cdots \cup D_n$, where $D_i (i = 1, 2, \cdots n)$ is a disk properly embedded in V_1 not ∂-parallel as illustrated in Figure 7.

Put $P = S \cap V_2$, then P is a planar surface properly embedded in V_2 with n boundary components. Suppose n is minimal among all such essential tori. If P is compressible

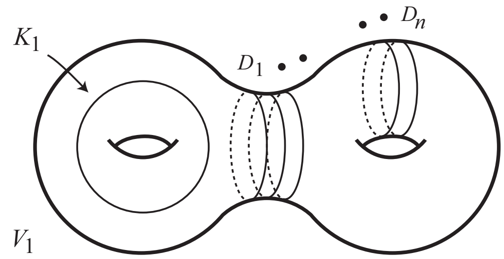


Figure 7 : $T \cap V_1$

in V_2. Then, by the minimality of n, there is a compressing disk for P intersecting K_2 in a single point. Then we see that L is composite and $C(1)$ holds by Theorem 3. Thus we may assume that P is incompressible in V_2. Let E be a meridian disk of V_2 with $E \cap K_2 = \emptyset$. Then, since ∂P consists of n components, $P \cap E \neq \emptyset$ and we may assume that each component of $P \cap E$ is an arc properly embedded in both P and E.

Let α be an outermost arc component of $P \cap E$ in E, and let Δ be the corresponding outer most disk. Then we can perform a boundary compression of P at α along Δ from V_2 to V_1, and we get a band, say b, in V_1, If b connects two different disks, then we can reduce the number of the components of $S \cap V_1$, and this contradicts the minimality of n. Thus b connects a single disk and we get an annulus $b \cup D_1$ or $b \cup D_n$ properly embedded in V_1. If the annulus is compressible, then there is a compressing disk for the annulus intersecting K_1 in a single point. Then we see that L is composite and $C(1)$ holds by Theorem 3. Thus we may assume that the annulus is incompressible, i.e., b winds around a handle at least once. We note that if a separating incompressible annulus winds around the handle not containing K_1 exactly once, then it is ∂-parllel and we can reduce the number n.

Then by [7, Lemma 3.4] and by the arguments similar to the proof of [8, Lemmata 1.1, 1.5] and [10, Theorem A], we have the following :

Lemma 9 *Under the above situations, by changing V_1 and V_2 if necessary, T can be isotoped into one of the following positions illustrated in Figure 8 :*

(1) *$T \cap V_1$ is a separating essential annulus winding around the handle not containing K_1 p times for some $|p| > 1$, and $T \cap V_2$ is a separating essential annulus winding around the handle not containing K_2 q times for some $|q| > 1$,*

(2) *$T \cap V_1$ is a separating essential annulus winding around the handle not containing K_1 p times for some $|p| > 1$, and $T \cap V_2$ is a separating essential annulus winding around the handle containing K_2 q times for some $|q| > 0$,*

(3) *$T \cap V_1$ is a separating essential annulus winding around the handle containing K_1 p times for some $|p| > 0$, and $T \cap V_2$ is a separating essential annulus winding around the handle containing K_2 q times for some $|q| > 0$,*

(4) $T \cap V_1$ consists of two separating essential annuli, one of them is winding around the handle containing K_1 p times for some $|p| > 0$, the other is a separating essential annulus winding around the handle not containing K_1 q times for some $|q| > 1$, and $T \cap V_2$ consists of two non-separating essential annuli winding the handle containing K_2 r times for some $|r| > 0$,
(5) $T \cap V_1$ consists of two non-separating essential annuli winding the handle containing K_1 p times for some $|p| > 0$, and $T \cap V_2$ consists of two non-separating essential annuli winding the handle containing K_2 q times for some $|q| > 0$.

We omitt the proof of this lemma, and by using this lemma we prove Theorem 6. If L is composite, then by Theorem 3 we have the condition $C(1)$. Hence we may assume that L is prime.

Suppose we are in Case (1). Let X_i be a genus two handlebody in V_i and Y_i a solid torus in V_i cut off by the annulus $T \cap V_i$ for $i = 1, 2$. Put $X = X_1 \cup X_2$ and $Y = Y_1 \cup Y_2$. Then X is a (1, 1)-knot exterior as illustrated in the left hand of Figure 9, and $Y = E(p, q)$. Since $|p| > 1$ and $|q| > 1$, Y is a non-trivial torus knot exterior, and hence X is a trivial knot exterior. Then we denote this trivial knot with the (1, 1)-decomposition by K_3 as illustrated in the right hand of Figure 9. Then by Lemma 8, $K_1 \cup K_2 \cup K_3$ is a link defined to construct an MS-link. Then by the construction, a meridian of the trivial knot K_3 is identified with a regular fiber of the Seifert fibered space $E(p, q)$. If at least one of $K_1 \cup K_3$ and $K_2 \cup K_3$ is a trivial link, then L is a composite link or a Hopf link, a contradiction. Hence both $K_1 \cup K_3$ and $K_2 \cup K_3$ are non-trivial links. Thus L is an MS-link and we have the condition $C(3)$.

Suppose we are in Case (2). Let X_i be a genus two handlebody in V_i and Y_i a solid torus in V_i cut off by the annulus $T \cap V_i$ for $i = 1, 2$. Put $X = X_1 \cup X_2$ and $Y = Y_1 \cup Y_2$. Then X is a (1, 1)-knot exterior as in case (1), and $Y = E(p, q)$ with $|p| > 1$ and $|q| > 0$. If $|q| = 1$, then $E(p, q)$ is a solid torus and K_2 is a core of the solid torus. Then T is a ∂-parallel torus in $E(L)$ and not essential. This contradiction implies $|q| > 1$, $E(p, q)$ is not a solid torus and K_2 is an exceptional fiber of the Seifert fibered space $E(p, q)$. This implies that X is a solid torus, i.e., X is a trivial knot exterior. By putting the trivial knot K_3, we have the link $K_1 \cup K_3$ defined to construct an MS-knot. In addition, a meridian of the trivial knot K_3 is identified with a regular fiber of the Seifert fibered space $E(p, q)$. Thus L is an union of an MS-knot and an exceptional fiber of the Seifert fibered space $E(p, q)$, and we have the condition $C(2)$.

Suppose we are in Case (3). Let X_i be a genus two handlebody in V_i and Y_i a solid torus in V_i cut off by the annulus $T \cap V_i$ for $i = 1, 2$. Put $X = X_1 \cup X_2$ and $Y = Y_1 \cup Y_2$. Then X is a (1, 1)-knot exterior as in case (1), and $Y = E(p, q)$ with $|p| > 0$ and $|q| > 0$. Since X contains neither K_1 nor K_2, the (1, 1)-knot is a non-trivial knot. If both $|p|$ and $|q|$ are greater than 1, then $E(p, q)$ is not a solid torus. This means T is

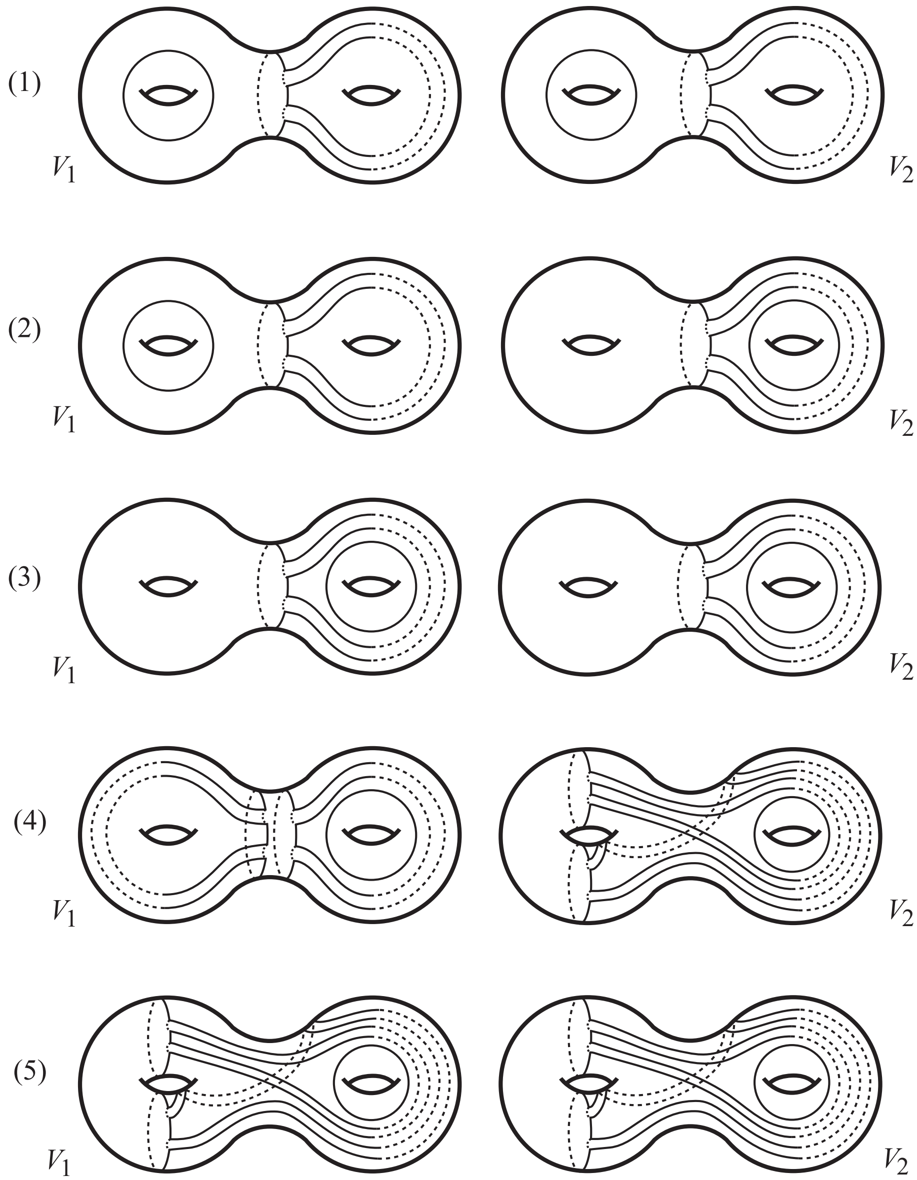


Figure 8 : Annuli in Heegaard splittings

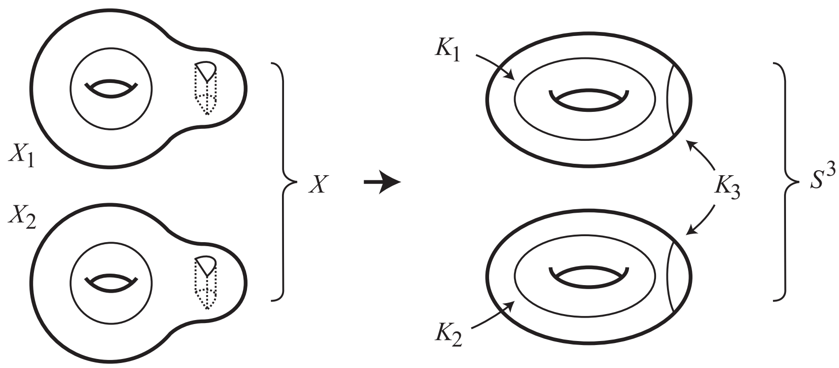


Figure 9 : (1, 1)-knot

an incompressibler torus in S^3, and this is a contradiction. Thus $|p| = 1$ or $|q| = 1$ and $E(p, q)$ is a solid torus. By the identification of X and Y, a meridian loop of the (1, 1)-knot is identified with a regular fiber of the Seifert fibration of $E(p, q)$, and this is a non-trivial Dehn surgery along the (1, 1)-knot. This means that a non-trivial Dehn surgery along a non-trivial knot yields S^3, and we have a contradiction by [3, Theorem 2], Thus case (3) doen not occur.

Suppse we are in case (4). Let X_1 be a genus two handlebody in V_1, and Y_1^1 and Y_1^2 two solid tori in V_1 cut off by the two annuli $T \cap V_1$. Let X_2 be a genus two handlebody in V_2 and Y_2 a solid torus in V_2 cut off by the annulus $T \cap V_2$. Put $X = X_1 \cup X_2$ and $Y = (Y_1^1 \cup Y_1^2) \cup Y_2$. Then X is a 2-bridge knot exterior as illustrated in Figure 10, and Y is a Seifert fibered space over a disk with three exceptional fibers whose indices are $|p| > 0$, $|q| > 1$ and $|r| > 0$. Since X contains neither K_1 nor K_2, the 2-bridge knot is a non-trivial knot. If $|p| > 1$ or $|r| > 1$, then Y is not a solid torus and T is an incompressible torus in S^3. This is a contradiction and $|p| = |q| = 1$. Then Y is a solid torus and this means that a non-trivial Dehn surgey along a non-trivial knot yields S^3. Then we have a contradiction by [3, Theorem 2], Thus case (4) doen not occur.

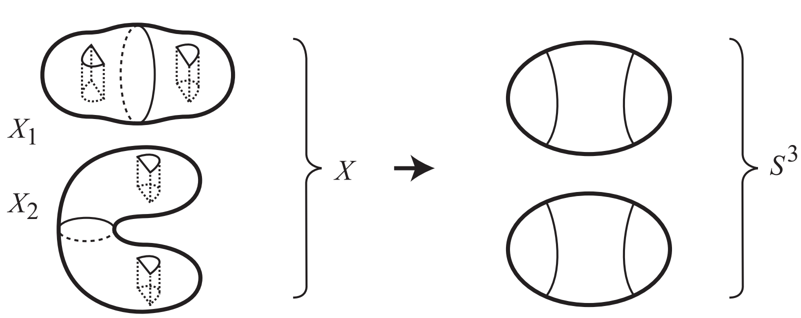


Figure 10 : 2-bridge knot

Suppose we are in case (5). Let X_i be a genus two handlebody in V_i and Y_i a solid torus in V_i cut off by the annulus $T \cap V_i$ for $i = 1, 2$. Put $X = X_1 \cup X_2$ and $Y = Y_1 \cup Y_2$. Then X is a 2-bridge knot exterior as in case (4), and Y is a Seifert fibered space over a Möbius band with 0, 1, or 2 exceptional fibers. This means that S^3 contains a Klein bottle, and this contradiction shows that case (5) does not occur.

On the other hand, if L satisfies one of the three conditions C(1), C(2) and C(3), then by tracing back the above arguments L has an essential torus in the exterior. This completes the proof of Theorem 6. □

4. Tangle decompositions

First we prepaire the following fact which is straightforward from the definition of (1, 1)-decompositions. So we omitt the proof.

Fact 10 *A knot K has a $(1,1)$-decomposition if and only if K is in ∂V for a standard genus two handlebody V in S^3 such that $K \cap D_1$ = a single point and $K \cap D_2$ = n points for some $n \geq 0$ as illustrated in Figure 11-(1), where D_1 and D_2 are meridian disiks of V, each of which has a cancelling disik in the complementary handlebody.*

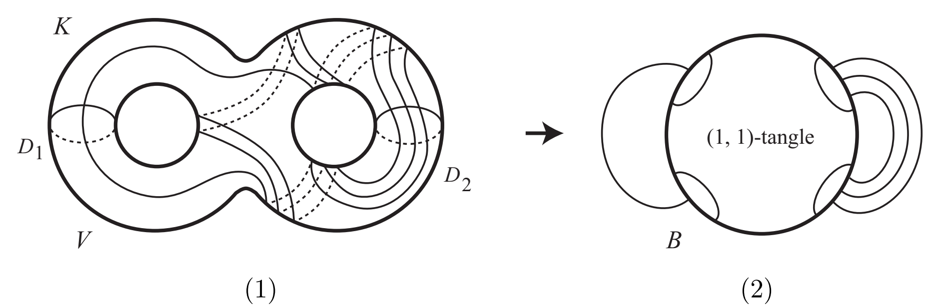


Figure 11 : (1, 1)-knot

In Figure 11-(1), by cutting open V with the disks D_1, D_2, we can get a 3-ball B and $n+1$ strings in ∂B as in Figure 11-(2). Conversely, let $(B, t_1 \cup t_2 \cup \cdots \cup t_n \cup t_{n+1})$ be an $n+1$-string tangle such that $t_1 \cup \cdots \cup t_{n+1}$ is parallel to ∂B and by closing the tangle with $n+1$ strings we can get a (1, 1)-knot in ∂V for a standard genus two handlebody V, then we call $(B, t_1 \cup t_2 \cup \cdots \cup t_n \cup t_{n+1})$ a (1, 1)-tangle as in Figure 11-(2). We note that if $n = 1$ then a (1, 1)-tangle is a rational tangle. Then we have :

Proposition 11 *Let K_1 be a non-trivial 2-bridge knot and K_2 a non-trivial (1, 1)-knot, and let $L = K_1 \cup K_2$ be a link illustrated in Figure 12-(1). Then L is a tunnel number one link of type II not of type I.*

Proof. Let γ be an arc in the 3-ball B_1 which connects the two strings of the rational tangle so that γ is a level arc of the 2-string trivial tangle. Then $K_1 \cup \gamma$ is deformed into a trivial glasses as in Figure 12-(2). This means that the complementary space of

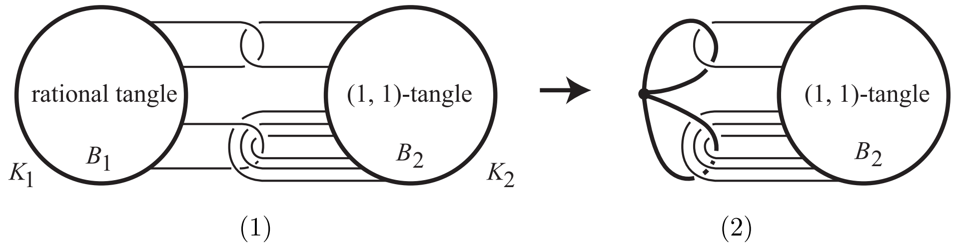


Figure 12 : $L = K_1 \cup K_2$

$K_1 \cup \gamma$ is a genus two handlebody, say V, and K_2 is in V. Then, since the (1, 1)-tangle in B_2 is parallel to ∂B_2 and since V can be regarded as a handlebody obtained by adding two 1-handles to B_2 as in Figure 11, we see that K_2 is parallel to ∂V and there is a cancelling meridian disk D intersecting K_2 in a single point. This shows that K_2 is a core of V and L is a tunnel number one link of type II. In addition, neither K_1 nor K_2 is a trivial knot. Then, by Theorem 1.5 of [2], L is not or type I. This completes the proof. □

For essential tangle decompositions of tunnel number one links obtained in the above proposition, we have :

Proposition 12 (1) *The link L in Proposition 11 has a 2-string essential tangle decomposition.* (2) *For any $n > 0$, there are infinitely many tunnel number one links of type II not of type I each of which has a 2-string essential tangle decomposition and an $n+1$-string essential tangle decomposition.*

Proof. (1) Decompose the link L in Proposition 11 into the two tangles with the line ① and with the line ② as in Figure 13-(1). Then the decomposition with the line ① is a 2-string essential tangle decomposition because K_1 and K_2 are non-trivial knots. (2) Consider the decomposition with the line ②, Then this makes an $n+1$-string tangle decomposition. However, it may not be essential. Consider the (1, 1)-knot K_2 illustrated in Figure 13-(2). Then we can see that the decomposition with the line ② is an $n+1$-string essential tangle decomposition. This completes the proof. □

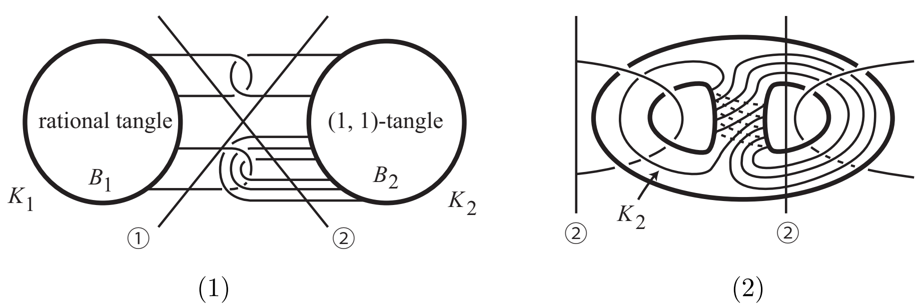


Figure 13 : Tangle decompositions

References

[1] C. McA. Gordon, On primitive sets of loops in the boundary of a handlebody. Topology Appl. Vol. 27, (1987) 285–299.

[2] C. McA. Gordon and A. W. Reid, Tangle decompositions of tunnel nuber one knots and links, J. Knot Theory and its Ramifications, Vol. 4, No. 3 (1995) 389–406.

[3] C. McA. Gordon and J. Luecke, Knots are determined by their complements. J. Amer. Math. Soc. Vol. 2 (1989) 371-415.

[4] J. Hempel, 3-manifolds, Annals of Mathematics Studies, No. 86 (1976) Princeton University Press

[5] K. Ishihara, On Heegaard splittings of link exteriors, Saitama Math. J. Vol. 25 (2008), 27-33.

[6] W. Jaco, Lectures on three manifold topology, CBMS Regional Conference Series in Mathematics No. 43 (1980)

[7] T. Kobayashi Structures of the Haken manifolds with Heegaard splittings of genus two. Osaka, J. Math., Vol. 21 (1984) 437–455.

[8] K. Morimoto, On minimum genus Heegaard splittings of some orientable closed 3-manifolds, Tokyo, J. Math., Vol. 12 (1989) 3321–355.

[9] K. Morimoto, On composite tunnel number one links, Topology and its Applications, Vol. 59 (1994) 59–71.

[10] K. Morimoto and M. Sakuma, On unknotting tunnels for knots, Mathematische Annalen, Vol. 289 (1991), 143-167.

[11] M. Eudave-Munoz and Y. Uchida, Non-simple links with tunnel number one, Proc. A. M. S. Vol. 124 (1996) 1567-1575.

[12] D. Rolfsen, Knots and Links, Mathematics Lecture Series No. 7 (1976) Publish or Perish Inc., Berkeley, Ca.