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2-Component links with genus two Heegaard splittings

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ABSTRACT

In the present paper, we consider two types of 2-component links with genus two Heegaard splittings. One of them is an ordinary tunnel number one link, and the other is a somewhat different tunnel number one link. We will try to detect the differences between those two types. In fact, we will characterize composite tunnel number one links of the second type, and tunnel number one links of the second type with essential tori.

Keywords: Genus two Heegaard splittings; tunnel number one links; connected sum; composite; essential tori; essential tangle decompositions.

Mathematics Subject Classification 2010: 57M25, 57M27

1. Introduction

Let K be a knot in S^3. Then, we say that K is tunnel number one if K is a non-trivial knot and there is a genus two Heegaard splitting (V_1, V_2) of S^3 such that K is a core of a handle of V_1 or of V_2.

Next, let $L = K_1 \cup K_2$ be a 2-component link in S^3, and let $N(L) = N(K_1) \cup N(K_2)$ be a regular neighborhood of L in S^3 and $E(L) = cl(S^3 - N(L))$ the exterior. Then $\partial E(L) = \partial N(K_1) \cup \partial N(K_2)$ is two tori. Suppose $E(L)$ has a genus two Heegaard splitting $E(L) = C_1 \cup C_2$, where $C_i (i = 1, 2)$ is a genus two handlebody or a genus two compression body. Then, we have the following two cases: one of them is that $\partial E(L)$ is contained in ∂C_1 or in ∂C_2, and the other is that, by changing the letters if necessary, $\partial N(K_1)$ ($\partial N(K_2)$ respectively) is contained in ∂C_1 (∂C_2 respectively).

So far, we say that L is tunnel number one if the former case occurs, and it seems that the latter case has not been much studied. See [3, 11, 2] for example. In knot case, there are no such ambiguities, but in link case, we need to consider the differences between these two cases. In the present paper, we study these two cases and try to detect the differences. So we will define the two types of tunnel number one links as follows.

We say that L is a tunnel number one link of type I if there is an arc γ in S^3 such that $L \cap \gamma = \partial\gamma$, γ connects K_1 and K_2 and the exterior $E(K_1 \cup \gamma \cup K_2)$ is a genus two handlebody (Fig. 1(a)), and that L is a tunnel number one link of type II if there is an arc γ in S^3 such that, by changing the letters if necessary, $L \cap \gamma = K_1 \cap \gamma = \partial\gamma$ and the exterior $E(K_1 \cup \gamma)$ is a genus two handlebody containing K_2 as a core of a handle (Fig. 1(b)).

In case of type I we call γ an unknotting tunnel of type I, and in case of type II we call γ an unknotting tunnel of type II. Put $V_1 = N(K_1 \cup \gamma \cup K_2)$ and $V_2 = cl(S^3 - V_1)$ in case of type I, and put $V_1 = N(K_1 \cup \gamma)$ and $V_2 = cl(S^3 - V_1)$ in case of type II. Then in both cases, (V_1, V_2) is a genus two Heegaard splitting of S^3 as in Fig. 1.

On inclusion relations of type I and type II, Ishihara showed in [6] the following:

Theorem 1 ([6, A part of Theorems 1.4, 1.5]). (1) *There are infinitely many tunnel number one links of type* I *not of type* II. (2) *There are infinitely many tunnel number one links of type* II *not of type* I.

By this theorem, we see that two families of tunnel number one links of type I and of type II are independent. Of course the intersection of these two families is not empty. For example, 2-bridge links are tunnel number one links of both types. In the present paper we characterize tunnel number one links of type II with some conditions.

First, we consider composite tunnel number one links. We say that a knot K is a 2-bridge knot if K is a non-trivial knot and there is a genus zero Heegaard splitting (B_1, B_2) of S^3 such that $K \cap B_i$ is a 2-string trivial arc system properly

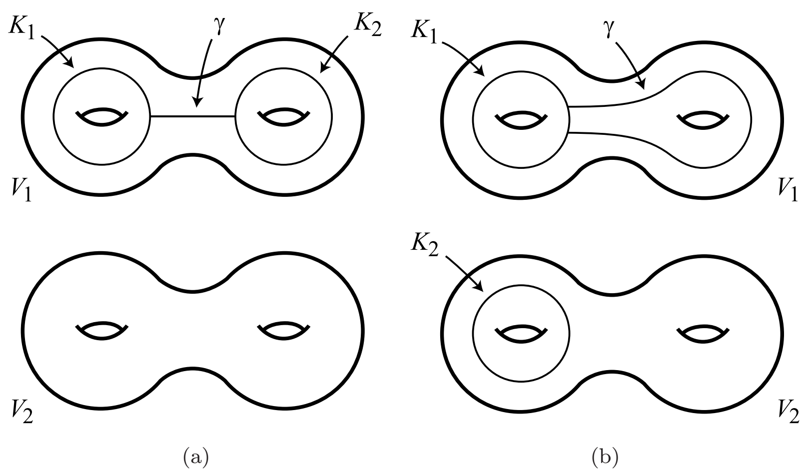


Fig. 1. Heegaard splittings and unknotting tunnels.

embedded in $B_i (i = 1, 2)$, and that a knot K is a (1, 1)-knot if K is a non-trivial knot and there is a genus one Heegaard splitting (V_1, V_2) of S^3 such that $K \cap V_i$ is a trivial arc properly embedded in $V_i (i = 1, 2)$. We call the decomposition a (1, 1)-decomposition. In [11], we characterized composite tunnel number one links of type I as follows:

Theorem 2 ([11, Theorem 1]). *Let L be a tunnel number one link of type* I. *Then, L is composite if and only if L is a connected sum of a* 2*-bridge knot and a Hopf link.*

In Sec. 2, we will characterize composite tunnel number one links of type II as follows:

Theorem 3. *Let L be a tunnel number one link of type* II. *Then, L is composite if and only if L is a connected sum of a* $(1, 1)$*-knot and a Hopf link.*

By these two theorems and the fact that 2-bridge knots are (1, 1)-knots, we see that the family of composite tunnel number one links of type I is properly contained in the family of composite tunnel number one links of type II, and that the difference is corresponding to the difference between 2-bridge knots and (1, 1)-knots.

Next, we consider tunnel number one links with essential tori, where a torus in the link exterior is essential if the torus is incompressible and is not ∂-parallel. Let $K_1 \cup K_2$ be a 2-bridge link not a Hopf link or a trivial link. Since K_2 is a trivial knot, $E(K_2)$ is a solid torus with $K_1 \subset E(K_2)$. Let $T(p, q)$ be a torus knot of type (p, q) for some relatively prime integers p, q with $|p| > 1$ and $|q| > 1$, and let $N(p, q)$ be a regular neighborhood of $T(p, q)$ and $E(p, q) = cl(S^3 - N(p, q))$ the exterior. Then, $E(p, q)$ is a Seifert fibered space over a disk with two exceptional points $D(-r/p, s/q)$ with $ps - qr = 1$. Let m be a longitude of the solid torus $E(K_2)$ in $\partial E(K_2)$ which is a meridian of K_2, and ℓ a regular fiber of $D(-r/p, s/q)$ in $\partial E(p, q)$, then we have an orientation preserving homeomorphism $f : E(K_2) \to N(p, q)$ with $f(m) = \ell$. Then we have a knot $f(K_1) \subset f(E(K_2)) = N(p, q) \subset S^3$ and call $f(K_1)$ an MS-knot. Then since $K_1 \cup K_2$ is not a Hopf link or a trivial link and since $T(p, q)$ is a non-trivial torus knot, MS-knot contains an essential torus in the exterior. Then in [11] we showed the following:

Theorem 4 ([12, Theorem A]). *Let K be a tunnel number one knot. Then, K has an essential torus in the exterior if and only if K is an MS-knot.*

For link version, we need to extend the definition of MS-knots as follows:

Let p, q be integers with $|p| = 1, |q| > 1$ or $|p| > 1, |q| = 1$, and let $E(p, q)$ be the exterior of the torus knot $T(p, q)$. In this case, $T(p, q)$ is a trivial knot and $E(p, q)$ is a solid torus. Then by the same way as above, we can define the knot $f(K_1)$ called an extended MS-knot and denoted by EMS-knot. Since $E(p, q)$ is a solid torus, EMS-knots do not have essential tori in the exteriors. However, the union of an EMS-knot and a regular fiber of $D(-r/p, s/q)$ is a link with an essential torus in the exterior since $|q| > 1$ or $|p| > 1$. Then as a link version of the above theorem, Eudave–Muñoz and Uchida showed the following:

Theorem 5 ([2, Theorems 1, 2]). *Let L be a tunnel number one link of type* I. *Then L has an essential torus in the exterior if and only if one of the following holds*:

$C(1)$: *L is a connected sum of a 2-bridge knot and a Hopf link*,
$C(2)$: *L is a union of an MS-knot and an exceptional fiber of the Seifert fibered space $D(-r/p, s/q)$, where $|p| > 1$ and $|q| > 1$*,
$C(3)$: *L is a union of an EMS-knot and a regular fiber of the Seifert fibered space $D(-r/p, s/q)$, where $|p| = 1, |q| > 1$ or $|p| > 1, |q| = 1$.*

Remark. In [2], tunnel number one links of type I with essential annuli have been classified. In fact, they noted in [2] that a tunnel number one link of type I with essential tori has essential annuli. Therefore, to state the above theorem, we need to pick up links with essential tori from [2, Theorems 1, 2]. Then [2, Theorem 1(i)], [2, Theorem 2(i)] and a subfamily of [2, Theorem 2(ii)] are all links we need. The situation has been stated in the note after the proof of [2, Theorem 2].

In Sec. 3, we will characterize tunnel number one links of type II with essential tori. Let's consider a 3-bridge link $K_1 \cup K_2 \cup K_3$ with the following conditions:

(i) $K_1 \cup K_2$ is a Hopf link,
(ii) both $K_1 \cup K_3$ and $K_2 \cup K_3$ are non-trivial 2-bridge links.

Then, since K_3 is a trivial knot, $E(K_3)$ is a solid torus with $K_1 \cup K_2 \subset E(K_3)$. Let m be a longitude of the solid torus $E(K_3)$ in $\partial E(K_3)$ which is a meridian of K_3, and let $T(p,q)$, $N(p,q)$, $E(p,q)$ and ℓ be as above, where $|p| > 1$ and $|q| > 1$. Then, we have an orientation preserving homeomorphism $f : E(K_3) \to N(p,q)$ with $f(m) = \ell$. Then we have a link $f(K_1 \cup K_2) \subset f(E(K_3)) = N(p,q) \subset S^3$ and call $L = f(K_1 \cup K_2)$ an MS-link. Then by the above condition (ii) and that $T(p,q)$ is a non-trivial knot, MS-link contains an essential torus in the exterior, and is not a composite link. The link illustrated in Fig. 2 is an example of an MS-link.

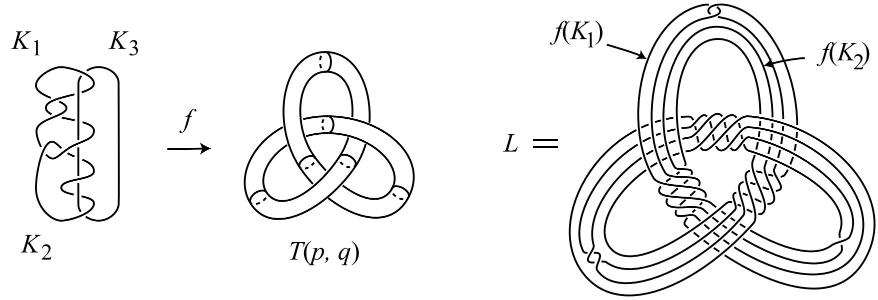


Fig. 2. MS-link.

Let K be a tunnel number one knot, and let (V_1, V_2) be a genus two Heegaard splitting such that V_1 contains K as a core of a handle. Then, K is isotopic to a loop in ∂V_1, and we denote the loop by the same notation K. Then, K is a loop in ∂V_2 too. Suppose K is primitive in V_2, i.e. K is isotopic to a core of a handle of V_2. According to [9] we call such a knot doubly primitive knot. Let A be an annulus in $\partial V_1 = \partial V_2$ such that a component of ∂A is K, and put $K' = \partial A - K$. Then, we call the 2-component link $K \cup K'$ in S^3 a DP-link.

Then, we show the following:

Theorem 6. *Let L be a tunnel number one link of type* II. *Then, L has an essential torus in the exterior if and only if one of the following holds*:

$C(1)$: *L is a connected sum of a $(1,1)$-knot and a Hopf link,*
$C(2)$: *L is a union of an MS-knot and an exceptional fiber of the Seifert fibered space $D(-r/p, s/q)$, where $|p| > 1$ and $|q| > 1$,*
$C(3)$: *L is a union of an EMS-knot and a regular fiber of the Seifert fibered space $D(-r/p, s/q)$, where $|p| = 1, |q| > 1$ or $|p| > 1, |q| = 1$,*
$C(4)$: *L is an MS-link*
$C(5)$: *L is a DP-link.*

By these two theorems, we see that the family of tunnel number one links of type I with essential tori is properly contained in the family of tunnel number one links of type II with essential tori. In fact, $C(2) \cup C(3)$ and $C(4) \cup C(5)$ are disjoint, because two components of the links in $C(2) \cup C(3)$ are separated by the essential tori, but two components of the links in $C(4) \cup C(5)$ are in one side of the essential tori.

Finally, we consider tunnel number one links with essential tangle decompositions. We say that a link L has an n-string essential tangle decomposition for $n > 0$ if there is a genus zero Heegaard splitting (B_1, B_2) of S^3 such that $(B_i, L \cap B_i)$ is an n-string essential tangle for both $i = 1, 2$, where $(B_i, L \cap B_i)$ is essential if $\partial B_i - L$ is incompressible in $B_i - L$ for $n > 1$ and $L \cap B_i$ is not a trivial arc in B_i for $n = 1$. Then Gordon and Reid showed in [3] the following:

Theorem 7 ([3, A part of Theorem 1.5]). *If a tunnel number one link L of type I has an essential tangle decomposition, then at least one of the two components of L is a trivial knot.*

On tangle decompositions of tunnel number one links of type II, Ishihara showed in [6] that 2-component Montesinos link $M(b; \frac{a_1}{b_1}, \frac{1}{2}, \frac{a_2}{b_2}, \frac{1}{2})$ is a tunnel number one link of type II not of type I, where (a_i, b_i) is a pair of relatively prime integers for $i = 1, 2$ with $|b_i| > 1$. Then, we see that this link has a 2-string essential tangle decomposition, and that both components of this link are 2-bridge knots.

In Sec. 4, we will show that there are infinitely many tunnel number one links of type II not of type I with n-string essential tangle decomposition for any $n > 0$, each of which consists of a 2-bridge knot and a (1, 1)-knot.

We note that Ishihara showed in [6] that if a tunnel number one link of type I has a trivial component then it is of type II too ([6, Theorem 1.7]). Thus by combining these theorems and examples, we see that the family of tunnel number one links of type I with essential tangle decompositions is properly contained in the family of tunnel number one links of type II with essential tangle decompositions.

So far, we have considered tunnel number one links with three conditions : composite, with essential tori and with essential tangle decompositions, and have seen the differences between tunnel number one links of type I and of type II. Although Theorem 1 says that there are infinitely many tunnel number one links of type I not of type II, we cannot get concrete examples of such links yet. So we set up the following problems at the end of Introduction.

Problems

(1) Construct concrete examples of tunnel number one links of type I not of type II.
(2) Show the type II version of Gordon–Reid's theorem (Theorem 7 above).

In the present paper, for standard terms and definitions in knot theory and 3-manifold topology, we refer to [5, 7, 14].

2. Proof of Theorem 3

Let $L = K_1 \cup K_2$ be a composite tunnel number one link of type II, and let S be a decomposing 2-sphere with $S \cap K_1 =$ two points and $S \cap K_2 = \emptyset$. Let γ be an unknotting tunnel of type II of L with $K_1 \cap \gamma = \partial\gamma$. Put $V_1 = N(K_1 \cup \gamma)$ and $V_2 = cl(S^3 - V_1)$, then (V_1, V_2) is a genus two Heegaard splitting of S^3 as illustrated in Fig. 1(b).

By $S \cap K_1 =$ two points, we may assume that $S \cap V_1 = D_1^* \cup D_2^* \cup D_1 \cup D_2 \cup \cdots \cup D_n$, where $D_i^* (i = 1, 2)$ is a meridian disk with $D_i^* \cap K_1 =$ one point and $D_j (j = 1, 2, \ldots n)$ is a disk properly embedded in V_1 not ∂-parallel as illustrated in Fig. 3.

Put $P = S \cap V_2$, then P is a planar surface properly embedded in V_2 with $n + 2$ boundary components. Suppose n is minimal among all such decomposing

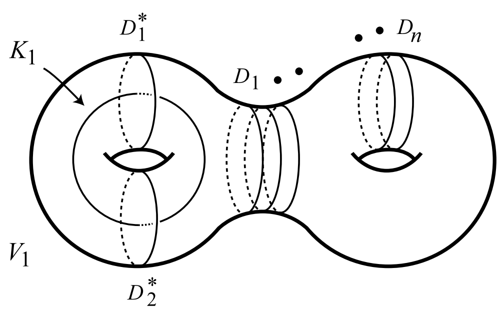


Fig. 3. $S \cap V_1$.

2-spheres. Then, we may assume that P is incompressible in $V_2 - K_2$. Let E be a meridian disk of V_2 with $E \cap K_2 = \emptyset$, and suppose $n > 0$. Then, since ∂P consists of $n+2(>2)$ components, $P \cap E \neq \emptyset$ and we may assume that each component of $P \cap E$ is an arc properly embedded in both P and E.

Let α be an outermost arc component of $P \cap E$ in E, and let Δ be the corresponding outer most disk. Then we can perform a boundary compression of P at α along Δ from V_2 to V_1, and we get a band, say b, in V_1.

Suppose b connects two different disks. If b connects D_1^* and D_2^*, then one of the two subarcs of K_1 cut off by $D_1^* \cup D_2^*$ is parallel into the disk $D_1^* \cup b_1 \cup D_2^*$. This means that one of the connected sum summands of L is a trivial knot. Thus, at least one of the two disks b connects is a disk D_i for some i. Then we can reduce the number of the components of $S \cap V_1$, and this contradicts the minimality of n.

Thus, b meets a single disk and α meets a single component of ∂P. If α cuts off a disk from P, then by standard cut and paste arguments, we can retake a meridian disk E with fewer components of $E \cap P$. Hence, α is an essential arc properly embedded in P. By this observation, we see that each component of $\partial(b \cup D_i)$ for some i is an essential loop in ∂V_1.

Suppose there is a non-separating disk in $D_1 \cup D_2 \cup \cdots \cup D_n$. Then, since each component of $V_1 - (D_1^* \cup D_2^* \cup D_1 \cup D_2 \cup \cdots \cup D_n)$ is a 3-ball, $b \cup D_i$ is a compressible annulus in V_1. Then, by performing a surgery along the compressing disk for $b \cup D_i$, we get a decomposing 2-sphere intersecting V_1 in fewer essential disks than n. This contradicts the minimality of n.

Thus $D_1 \cup D_2 \cup \cdots \cup D_n$ are all mutually parallel separating disks. Then, we may assume that b meets the separating disk D_n and $b \cup D_n$ is an incompressible annulus as in Fig. 4.

Suppose b winds around a handle of V_1 p times for some $p > 1$. Then, since $cl(S - (D_n \cup b))$ consists of two disks, the union of the solid torus cut off by $D_n \cup b$ and one of the two disks shows that S^3 contains a lens space summand of the order p. Hence, $p = 1$, and the annulus $D_n \cup b$ is a ∂-parallel annulus. Then, we can reduce the number of the components of $S \cap V_1$, and this contradicts the minimality

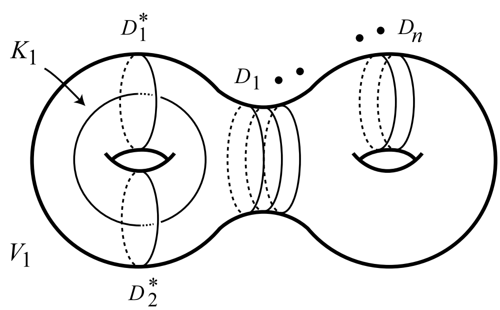


Fig. 4. $D_n \cup b$.

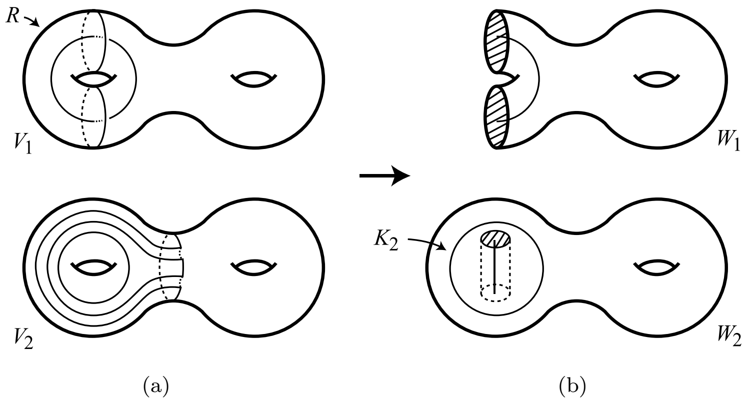


Fig. 5. $(V_1 \cup V_2) \to (W_1 \cup W_2)$.

of n. After all, we have $n = 0$ and $S \cap V_1 = D_1^* \cup D_2^*$. Thus, by [8, Lemma 3.2], $S \cap V_2$ is a separating annulus consisting of a separating disk and a band winding around a handle containing K_2 exactly once as in Fig. 5(a).

Let R be the 3-ball in V_1 cut off by $D_1^* \cup D_2^*$ indicated in Fig. 5(a). Put $W_1 = cl(V_1 - R)$ and $W_2 = V_2 \cup R$ as in Fig. 5(b). Then W_1 is a solid torus, and since the annulus $S \cap V_2$ winds around a handle containing K_2 once, W_2 is a solid torus too. Then, since $W_1 \cap K_1$ is a trivial arc in W_1 and $W_2 \cap K_1$ is a trivial arc in W_2, K_1 has a (1, 1)-decomposition. Moreover, K_2 is a trivial loop in W_2 bounding a disk intersecting K_1 in a single point. This shows that L is a connected sum of a (1, 1)-knot and a Hopf link.

On the other hand, the converse is proved by tracing back the above arguments, and this completes the proof of Theorem 3.

3. Proof of Theorem 6

Before the proof of Theorem 6, we prepare the following lemma:

Lemma 8. *Let $K_1 \cup K_2 \cup K_3$ be a 3-component link with the following conditions*:

(i) *there is a genus one Heegaard splitting (V_1, V_2) of S^3 such that K_i is a core of $V_i (i = 1, 2)$,*
(ii) *K_3 is a trivial knot,*
(iii) *K_3 intersects V_i in a trivial arc and there is a trivializing disk Δ_i for $K_3 \cap V_i$ with $\Delta_i \cap K_i = \emptyset$ $(i = 1, 2)$ as in Fig. 6.*

Then, $K_1 \cup K_2$ is a Hopf link and $K_1 \cup K_2 \cup K_3$ is a 3-bridge link.

Proof. By the condition (i), it is clear that $K_1 \cup K_2$ is a Hopf link.

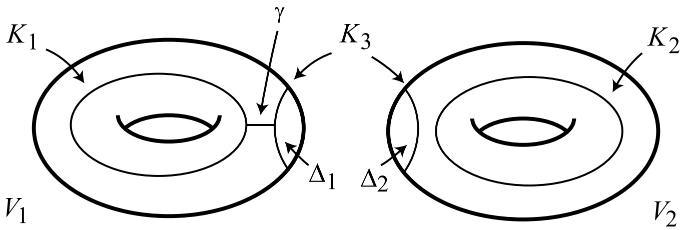


Fig. 6. $K_1 \cup K_2 \cup K_3$.

Claim *We can take a meridian disk* D_i $(i = 1, 2)$ *of* V_i *so that* $D_i \cap K_i$ *is a single point,* $D_i \cap K_3 = \emptyset$ *and* $\partial D_1 \cap \partial D_2$ *is a single point.*

Proof of Claim. Let γ be a "straight arc" connecting K_1 and $K_3 \cap V_1$ in V_1 as in Fig. 6 and put $\Gamma = K_1 \cup \gamma \cup (K_3 \cap V_1)$. Then, we can regard V_1 as a thin regular neighborhood of the graph Γ.

Let D be a disk with $\partial D = K_3$ by the triviality of K_3. Then by considering intersections $D \cap \Gamma$, we can put $D \cap V_1 = E_0 \cup E_1 \cup \cdots \cup E_m \cup F_1 \cup \cdots \cup F_k$, where E_0 is a disk with $\partial E_0 = (K_3 \cap V_1) \cup$ (an arc in ∂V_1), $E_i (i = 1, 2, \ldots, m)$ is a meridian disk of V_1 intersecting K_1 in a single point and $F_j (j = 1, 2, \ldots k)$ is a disk intersecting γ in a single point (see Fig. 7(a) or 7(b)). In this situation, we may assume that $m + k$ is minimal among all such disks. If $V_1 \cap D = E_0$, then $D \cap V_2$ is a single disk too, and by using these disks we can take the required meridian disks. So we may assume that $m > 0$ or $k > 0$.

Put $P = V_2 \cap D = cl(D - (E_0 \cup E_1 \cup \cdots \cup E_m) - (F_1 \cup \cdots \cup F_k))$. Then P is a planar surface such that $\partial P = (K_3 \cap V_2) \cup (E_0 \cap \partial V_2) \cup \partial(E_1 \cup \cdots \cup E_m) \cup \partial(F_1 \cup \cdots \cup F_k)$, where $\partial E_i (i = 1, 2, \ldots, m)$ is a longitude of V_2. Let G be a meridian disk of V_2. Then, by the existence of Δ_2, we may assume that $G \cap K_3 = \emptyset$, $G \cap K_2 =$ a single point.

Suppose $m = 0$. Then $k > 0$ because $m > 0$ or $k > 0$. If $G \cap P = \emptyset$, then ∂G is a longitude in ∂V_1 not intersecting $D \cap \partial V_1$. Let R be a disk in ∂V_1 bounded by ∂F_k, then we have $D \cap \partial V_1 \subset R$ and $R \cap \partial G = \emptyset$. Then, we can take a meridian disk H

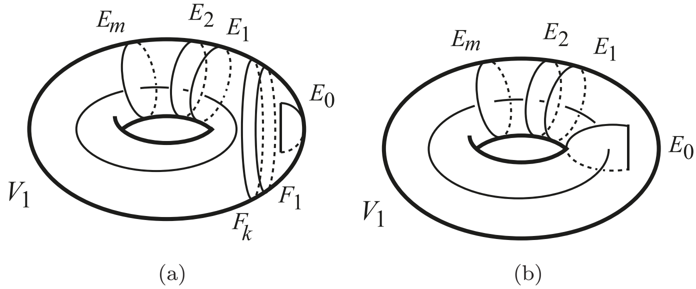


Fig. 7. $D \cap V_1$.

in V_1 such that $H \cap K_1 =$ a single point and $\partial H \cap \partial G =$ a single point. Moreover we may assume that $H \cap K_3 = \emptyset$ because we can take H so that $H \cap R = \emptyset$. Then, G and H are the required meridian disks.

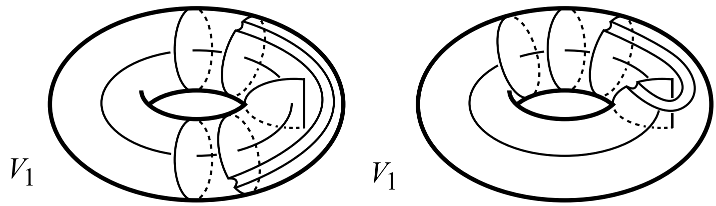
Hence, we may assume that $G \cap P \neq \emptyset$ and that each component of $G \cap P$ is a loop or an arc properly embedded in G. Suppose there is a loop component, say ℓ, in $G \cap P$, and let G_1 be a disk in G with $\partial G_1 = \ell$. Since ℓ is a loop in D, ℓ bounds a disk in D. Then, by standard cut and paste arguments, we can retake the disk D to eliminate the intersection loop ℓ. Hence, we may assume that each component of $G \cap P$ is an arc properly embedded in G. Then, we can find an outermost arc component α_1 of $G \cap P$ in G and the corresponding outermost disk δ_1 in G with $\delta_1 \cap K_2 = \emptyset$ because $G \cap K_2 =$ a single point. Then, we can perform a boundary compression of P at α_1 along δ_1 from V_2 to V_1, and we get a band, say b_1, in V_1.

If b_1 connects the different components F_i and F_j, then we can reduce the number of the disks. Hence, we may assume that b_1 meets F_k and $F_k \cup b_1$ is an annulus, say A_1. If A_1 is a compressing annulus in V_1, then we have a compressing disk for A_1 which intersects K_1 in a single point. Then by cutting D by the compressing disk, we can retake D so that $D \cap V_1$ consists of fewer disks than $m + k$. Hence, A_1 is an annulus winding around the longitude of V_1 at least once. By repeating boundary compressions from V_2 to V_1, we have $D \cap V_1 = A_1 \cup A_2 \cup \cdots \cup A_k \cup E_0$, where A_i $(i = 1, 2, \ldots, k)$ are all mutually parallel incompressible annuli. In this situation, $D \cap V_2$ consists of k annuli and a single disk, say Q. Then, since a component of ∂Q is identified with a component of ∂A_i for some i, Q is a meridian disk of V_2 and A_i winds around V_1 exactly once. Let H be a meridian disk of V_1 intersecting a component of ∂A_i in a single point and K_1 in a single point with $H \cap K_3 = \emptyset$. Then, ∂H intersects ∂Q in a single point, and since ∂Q and ∂G are isotopic to each other in ∂V_2 and we can take the isotopy not intersecting $K_3 \cap \partial V_2$, we may assume that ∂H intersects ∂G in a single point. Thus G and H are the required meridian disks. Hence, hereafter we assume $m > 0$.

Suppose $E_0 \cap K_1 = \emptyset$ as in Fig. 7(a). If b_1 meets E_1 or E_m, say E_1, then the annulus $E_1 \cup b_1$ is compressing and, by using the compressing disk, we can retake the disk D so that $D \cap V_1$ consists of fewer disks than $m + k$. This contradicts the minimality of $m + k$. If b_1 meets F_k, then we have a similar contradiction. Hence b_1 connects two different disks. Then we have a similar contradiction. Thus we may assume that $E_0 \cap K_1 \neq \emptyset$, $k = 0$ and $D \cap V_1 = E_0 \cup E_1 \cup \cdots \cup E_m$ as in Fig. 7(b).

Suppose b_1 meets E_1 or E_m, say E_1, and suppose a component of $\partial(E_1 \cup b_1)$ bounds a disk in ∂V_1 containing the two points $K_3 \cap \partial V_1$. In this case, by using the compressing disk for the annulus $E_1 \cup b_1$, we can retake D so that $D \cap V_1$ consists of E_0 and at most m meridian disks and b_1 connects E_1 and E_m. Hence, we may assume that b_1 connects two different components, and we have the following two cases:

(1) b_1 connects E_1 and E_m and $\partial(E_1 \cup b_1 \cup E_m)$ bounds a disk in ∂V_1 containing the two points $K_3 \cap \partial V_1$ as in Fig. 8(a).



(a) (b)

Fig. 8. b_1 in V_1.

(2) b_1 connects E_0 and E_1 or E_m, say E_1, and there is no such a disk R that ∂R consists of a subarc of $E_0 \cup b_1 \cup E_1$ and a subarc of K_1 with $D \cap \mathrm{Int} R = \emptyset$ as in Fig. 8(b).

Suppose we are in Case (1). Let α_2 be an outermost arc of $G \cap P$ in G at the second stage, and let δ_2 and b_2 be the corresponding outermost disk and the band. If b_2 meets a single component E_i, then, by the same arguments as above, we can retake D so that b_2 can be regarded as a band connecting two different components. Hence, we may assume that b_2 connects E_i and E_j for $i \neq j$. Suppose $\delta_1 \cap \delta_2 = \emptyset$. If there is a disk R such that ∂R consists of a subarc of $E_i \cup b_2 \cup E_j$ and a subarc of K_1 with $D \cap \mathrm{Int} R = \emptyset$, then by changing the order of b_1 and b_2 and by using the disk R we can reduce the number of the disks $D \cap V_1$. If b_2 connects E_1 and E_m and b_2 and b_1 are parallel, then a component of $\partial(E_1 \cup b_1 \cup b_2 \cup E_m)$ bounds a disk in ∂V_1 contains no points of $K_3 \cap \partial V_1$. Then by using this disk we can retake the disk D with fewer components of $D \cap V_1$. If b_2 connects E_0 and E_1, then by the deformation as in Fig. 9, we may assume that b_2 is a “straight” band and we can take a disk R as above. Hence, we may assume that $\delta_1 \cap \delta_2 \neq \emptyset$, in fact we have $\delta_1 \subset \delta_2$.

By continuing these arguments for $b_3, b_4, \ldots$, we may assume that all components of $G \cap P$ are parallel and b_i $(i = 2, 3, 4, \ldots)$ runs through over b_{i-1}.

First suppose $m = 2$. Then, since b_2 runs through over b_1, b_2 connects E_1 and E_2 as in Fig. 10(a). Then, each component of $\partial(E_1 \cup b_1 \cup b_2 \cup E_2)$ winds around the longitude of V_1 at least two times. However, at least one component bounds

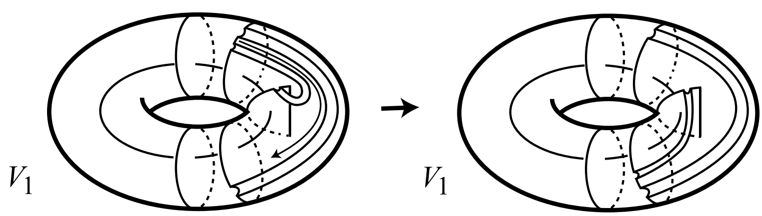


Fig. 9. Deformation of b_2 along b_1.

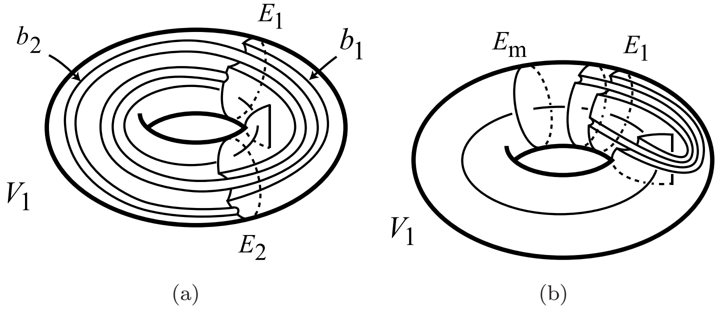


Fig. 10. $b_1, b_2, \cdots, b_{m-1}$ in V_1.

a disk in V_2, because P consists of an annulus and a disk at this stage. This is a contradiction.

In general case, let $b_1, b_2, \ldots, b_k$ be the bands produced by the boundary compressions, where b_1 connects E_1 and E_m, b_2 connects E_2 and $E_{m-1}, \ldots, b_k$ connects E_k and E_{m-k+1}. In this case we may assume $m = 2k$. Then as in the case of $m = 2$, let $b_{k+1}, b_{k+2}, \ldots, b_{2k}$ be the bands produced by the boundary compressions, where b_{k+1} connects E_k and $E_{k+1}, \ldots, b_{2k}$ connects E_1 and E_{2k}. Then, each component of ∂P at this stage is an essential loop in ∂V_1 winding around the longitude of V_1 at least two times. However, at this stage, P has at least one disk component. This is a contradiction. Afterall, we see that Case (1) does not occur.

Next suppose we are in Case (2). By the arguments similar to Case (1), we may assume that all components of $G \cap P$ are parallel as in Case (1).

Perform the sequence of boundary compressions at $\alpha_1, \alpha_2, \ldots \alpha_{m-1}$. Then, the sequence of bands $b_1, b_2, \cdots b_{m-1}$ are produced as in Fig. 10(b), where b_1 connects E_0 and E_1, b_2 connects E_1 and $E_2, \ldots, b_{m-1}$ connects E_{m-2} and E_{m-1}. At the $(m-1)$th stage, P is an annulus and ∂P consists of ∂E_m and an arc connecting the two points $K_3 \cap \partial V_2$. Then, ∂E_m intersects G in a single point. Thus by taking E_m as D_1 and by taking G as D_2, we can complete the proof of the claim. □

Let $N(D_i)$ be a regular neighborhood of D_i in V_i $(i = 1, 2)$. If $D_i \cap \Delta_i \neq \emptyset$, then by standard cut and paste arguments we can retake Δ_i so that $D_i \cap \Delta_i = \emptyset$. Put $W_1 = cl(V_1 - N(D_1)) \cup N(D_2), W_2 = cl(V_2 - N(D_2)) \cup N(D_1)$. Then, by the above claim, both W_1 and W_2 are 3-balls and (W_1, W_2) is a genus zero Heegaard splitting of S^3. Put $K_1 \cap W_i = \alpha_i$, $K_2 \cap W_i = \beta_i$, and $K_3 \cap W_i = \gamma_i$ $(i = 1, 2)$. Then, since $cl(V_1 - N(D_1)) \cap N(D_2)$ is a disk, a trivializing disk for α_1 in $cl(V_1 - N(D_1))$ and Δ_1 extend to trivializing disks for α_1 and γ_1 in W_1 disjoint from a trivializing disk for β_1. Hence, $(W_1, \alpha_1 \cup \beta_1 \cup \gamma_1)$ is a 3-string trivial tangle, and by the same arguments $(W_2, \alpha_2 \cup \beta_2 \cup \gamma_2)$ is a 3-string trivial tangle too. Thus $K_1 \cup K_2 \cup K_3$ is a 3-bridge link, and this completes the proof of Lemma 8.

Now, we prove Theorem 6. Let $L = K_1 \cup K_2$ be a tunnel number one link of type II with an essential torus, and let T be the essential torus in the exterior. Let γ be an unknotting tunnel of type II of L with $K_1 \cap \gamma = \partial\gamma$, and put $V_1 = N(K_1 \cup \gamma)$ and $V_2 = cl(S^3 - V_1)$. Then, (V_1, V_2) is a genus two Heegaard splitting of S^3 as illustrated in Fig. 1(b).

By considering the intersections of T and $K_1 \cup \gamma$, we may assume that $T \cap V_1 = D_1 \cup D_2 \cup \cdots \cup D_n$, where $D_i (i = 1, 2, \ldots n)$ is a disk properly embedded in V_1 not ∂-parallel as illustrated in Fig. 11.

Put $P = T \cap V_2$, then P is a genus one surface properly embedded in V_2 with n boundary components. Suppose n is minimal among all such essential tori. Then by the minimality of n, P is incompressible in $V_2 - K_2$. Let E be a meridian disk of V_2 with $E \cap K_2 = \emptyset$. Then, we may assume that each component of $P \cap E$ is an arc properly embedded in both P and E.

Let α be an outermost arc component of $P \cap E$ in E, and let Δ be the corresponding outer most disk. Then, we can perform a boundary compression of P at α along Δ from V_2 to V_1, and we get a band, say b, in V_1. If b connects two different disks, then we can reduce the number of the components of $T \cap V_1$, and this contradicts the minimality of n. Thus b meets a single disk and we get an annulus $b \cup D_1$ or $b \cup D_n$ properly embedded in V_1, because the annulus is incompressible in $V_1 - K_1$. If the annulus is compressible in V_1, then there is a compressing disk for the annulus intersecting K_1 in a single point. Then, we see that L is composite and $C(1)$ holds by Theorem 3. Thus we may assume that the annulus is incompressible in V_1, i.e. b winds around a handle at least once. We note that if a separating incompressible annulus winds around the handle not containing K_1 exactly once, then it is ∂-parallel and we can reduce the number n.

Then, by [8, Lemma 3.4] and by the arguments similar to the proof of [10, Lemmata 1.1 and 1.5] and [12, Theorem A], we have the following:

Lemma 9. *Under the above situations, by changing V_1 and V_2 if necessary, T can be isotoped into one of the following positions illustrated in Fig.* 12:

(1) *$T \cap V_1$ is a separating incompressible annulus winding around the handle not containing K_1 p times for some $|p| > 1$, and $T \cap V_2$ is a separating incompressible annulus winding around the handle not containing K_2 q times for some $|q| > 1$,*

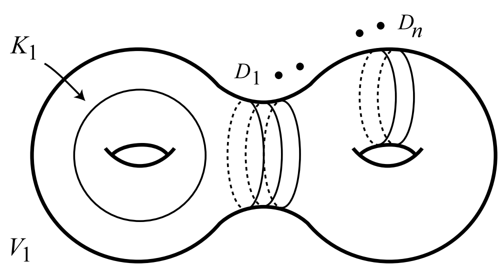


Fig. 11. $T \cap V_1$.

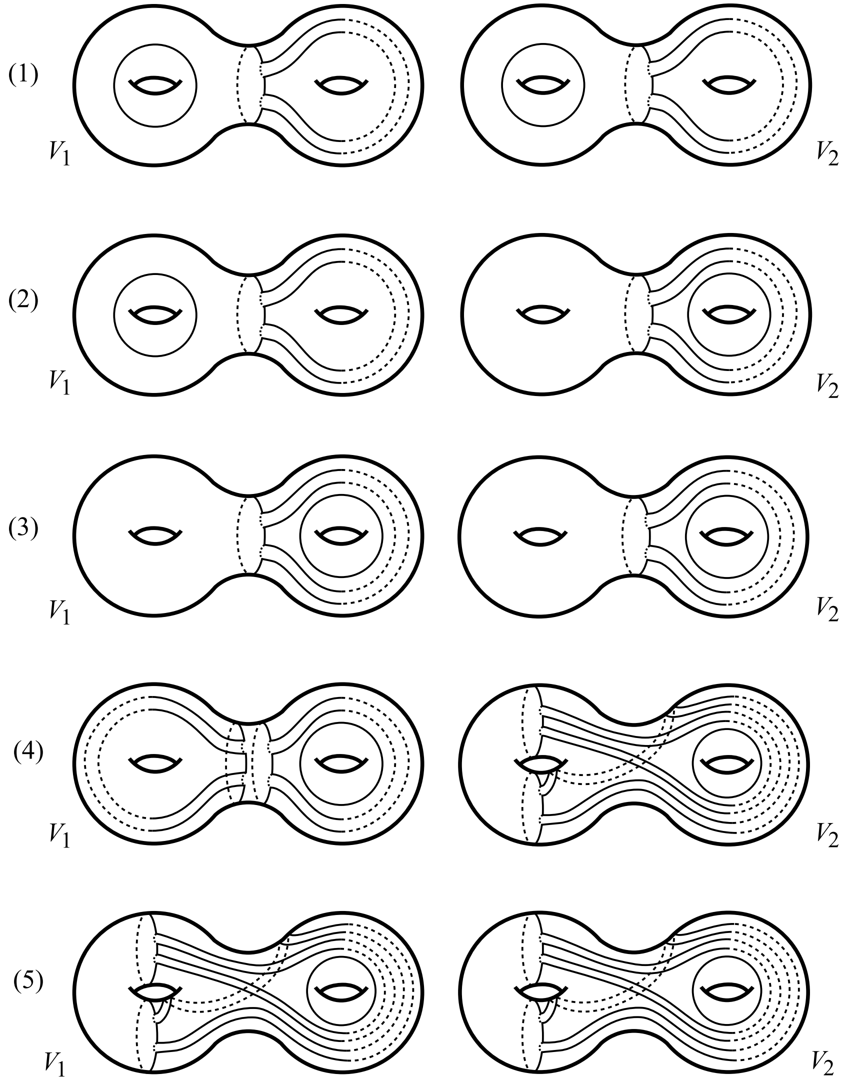


Fig. 12. Annuli in handlebodies.

(2) *$T \cap V_1$ is a separating incompressible annulus winding around the handle not containing K_1 p times for some $|p| > 1$, and $T \cap V_2$ is a separating incompressible annulus winding around the handle containing K_2 q times for some $|q| > 0$,*

(3) *$T \cap V_1$ is a separating incompressible annulus winding around the handle containing K_1 p times for some $|p| > 0$, and $T \cap V_2$ is a separating incompressible annulus winding around the handle containing K_2 q times for some $|q| > 0$,*

(4) *$T \cap V_1$ consists of two separating incompressible annuli, one of them is winding around the handle containing K_1 p times for some $|p| > 0$, the other is winding around the handle not containing K_1 q times for some $|q| > 1$, and $T \cap V_2$ consists of two non-separating incompressible annuli winding the handle containing K_2 r times for some $|r| > 0$,*
(5) *$T \cap V_1$ consists of two non-separating incompressible annuli winding the handle containing K_1 p times for some $|p| > 0$, and $T \cap V_2$ consists of two non-separating incompressible annuli winding the handle containing K_2 q times for some $|q| > 0$.*

We omit the proof of this lemma, and by using this lemma we prove Theorem 6. If L is composite, then by Theorem 3 we have the condition $C(1)$. Hence, we may assume that L is prime.

Suppose we are in Case (1). Let X_i be a genus two handlebody in V_i and Y_i a solid torus in V_i cut off by the annulus $T \cap V_i$ for $i = 1, 2$. Put $X = X_1 \cup X_2$ and $Y = Y_1 \cup Y_2$. Then, X is a (1, 1)-knot exterior in some lens space as illustrated in the left-hand of Fig. 13, and $Y = E(p, q)$. Since $|p| > 1$ and $|q| > 1$, Y is a non-trivial torus knot exterior, and hence X is a solid torus. Since $X \cup Y$ is S^3, a regular fiber of the Sefert fibered space $Y = E(p, q)$ is identified with a longitude of the solid torus X and it is a meridian of the (1, 1)-knot. Hence, the lens space is S^3.

Then we denote the (1, 1)-knot in S^3 with the (1, 1)-decomposition by K_3 as illustrated in the right-hand of Fig. 13. Then K_3 is a trivial knot in S^3 because X is a solid torus, and by Lemma 8, $K_1 \cup K_2 \cup K_3$ is a 3-bridge link. If at least one of $K_1 \cup K_3$ and $K_2 \cup K_3$ is a trivial link, then L is a composite link or a Hopf link, and this is a contradiction. Hence, both $K_1 \cup K_3$ and $K_2 \cup K_3$ are non-trivial 2-bridge links. Thus, L is an MS-link and we have the condition $C(4)$.

Suppose, we are in Case (2). Let X_i be a genus two handlebody in V_i and Y_i a solid torus in V_i cut off by the annulus $T \cap V_i$ for $i = 1, 2$. Put $X = X_1 \cup X_2$ and

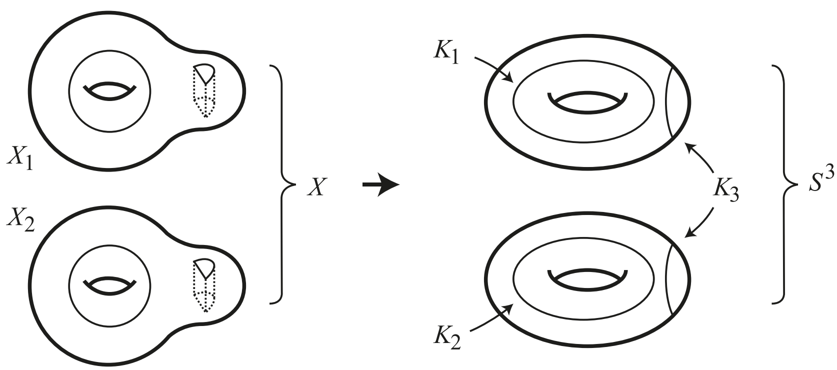


Fig. 13. (1, 1)-knot.

$Y = Y_1 \cup Y_2$. Then, X is a (1, 1)-knot exterior in some lens space as in Case (1), and $Y = E(p,q)$ with $|p| > 1$ and $|q| > 0$.

If $|q| > 1$, then $E(p,q)$ is a non-trivial torus knot exterior and K_2 is an exceptional fiber of the Seifert fibered space $E(p,q)$. Then, as in Case (1), X is a solid torus, the lens space is S^3 and the (1, 1)-knot is a trivial knot in S^3. By putting the trivial knot K_3 and by [12, Lemma 2.3], we have the link $K_1 \cup K_3$ defined to get an MS-knot. In addition, a meridian of the trivial knot K_3 is identified with a regular fiber of the Seifert fibered space $E(p,q)$. Thus, L is an union of an MS-knot and an exceptional fiber of the Seifert fibered space $E(p,q)$, and we have the condition $C(2)$.

If $|q| = 1$, then $E(p,q)$ is a solid torus and K_2 is a regular fiber of the Seifert fibration of the solid torus. By using Cyclic Surgery Theorem of [1] and by [13], X is the Seifert fibered space over a disk with one or two exceptional fibers.

If it has one exceptional fiber, then X is a solid torus, i.e. X is a trivial knot exterior of S^3. Then by the same arguments as above, we see that L is an union of an EMS-knot and a regular fiber of $E(p,q)$, and we have the condition $C(3)$. If it has two exceptional fibers, then the (1, 1)-knot is a non-trivial torus knot which is not a core of the lens space. Then by [10, Theorem 3], the (1, 1)-knot has a unique (1, 1)-decomposition and we can perform boundary compression from V_1 to V_2 and from V_2 to V_1 simultaneously. Then we have the same situation as the case of $|q| > 1$ because X has two exceptional fibers. Thus, we have the condition $C(2)$.

Suppose, we are in Case (3). Let X_i be a genus two handlebody in V_i and Y_i a solid torus in V_i cut off by the annulus $T \cap V_i$ for $i = 1, 2$. Put $X = X_1 \cup X_2$ and $Y = Y_1 \cup Y_2$. Then, X is a (1, 1)-knot exterior in some lens space, and $Y = E(p,q)$ with $|p| > 0$ and $|q| > 0$. Since X contains neither K_1 nor K_2, X is not a solid torus. If $|p| > 1$ and $|q| > 1$, then $E(p,q)$ is not a solid torus and T is incompressible torus in S^3. This is a contradiction. Suppose $|p| > 1$ and $|q| = 1$ or $|p| = 1$ and $|q| > 1$, then by the same arguments as the proof of Case (2), X is a Seifert fibered space over a disk with one or two exceptional fibers. If it has one exceptional fiber, then X is a solid torus and this is a contradiction. If it has two exceptional fibers, then by the same arguments as the proof of Case (2), we can perform boundary compressions from V_1 to V_2 and from V_2 to V_1 simultaneously. Then, we have the same situation as the Case (1) and we have the condition $C(4)$.

Suppose $|p| = |q| = 1$. Then, K_1 is isotopic to a component of $\partial(T \cap V_1)$ in V_1 and K_2 is isotopic to a component of $\partial(T \cap V_2)$ in V_2. This means K_1 and K_2 are two copies of a doubly primitive knot in $\partial V_1 = \partial V_2$. Hence, L is a DP-link and we have the condition $C(5)$.

Suppose, we are in Case (4). Let X_1 be a genus two handlebody in V_1, and Y_1^1 and Y_1^2 two solid tori in V_1 cut off by the two annuli $T \cap V_1$. Let X_2 be a genus two handlebody in V_2 and Y_2 a solid torus in V_2 cut off by the annulus $T \cap V_2$. Put $X = X_1 \cup X_2$ and $Y = (Y_1^1 \cup Y_1^2) \cup Y_2$. Then, X is a 2-bridge knot exterior as illustrated in Fig. 14, and Y is a Seifert fibered space over a disk with three

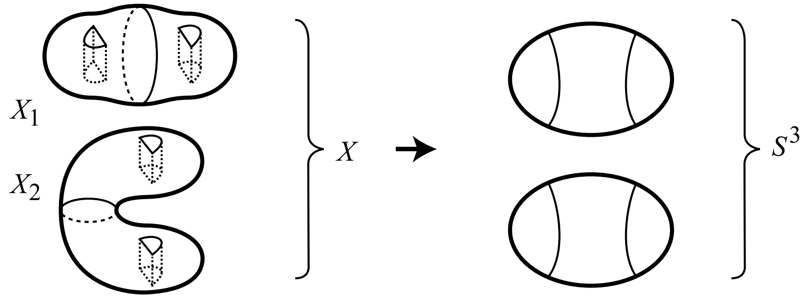


Fig. 14. 2-bridge knot.

exceptional fibers whose indices are $|p| > 0$, $|q| > 1$ and $|r| > 0$. Since X contains neither K_1 nor K_2, the 2-bridge knot is a non-trivial knot. If $|p| > 1$ or $|r| > 1$, then Y is not a solid torus and T is an incompressible torus in S^3. This is a contradiction, and hence $|p| = |r| = 1$. Then, Y is a solid torus and this means that a non-trivial Dehn surgery along a non-trivial knot yields S^3. Then, we have a contradiction by [4, Theorem 2], Thus Case (4) does not occur.

Suppose we are in Case (5). Let X_i be a genus two handlebody in V_i and Y_i a solid torus in V_i cut off by the annulus $T \cap V_i$ for $i = 1, 2$. Put $X = X_1 \cup X_2$ and $Y = Y_1 \cup Y_2$. Then X is a 2-bridge knot exterior and Y is a Seifert fibered space over a Möbius band with 0, 1, or 2 exceptional fibers. This means that S^3 contains a Klein bottle. This contradiction shows that Case (5) does not occur.

On the other hand, if L satisfies one of the five conditions $C(1)$, $C(2)$, $C(3)$, $C(4)$ and $C(5)$, then by tracing back the above arguments L has an essential torus in the exterior. This completes the proof of Theorem 6.

4. Tangle Decompositions

First we prepare the following fact which is straightforward from the definition of (1, 1)-decompositions. So we omit the proof.

Fact 10. A knot K has a $(1, 1)$-decomposition if and only if K is in ∂V for a standard genus two handlebody V in S^3 such that $K \cap D_1 =$ a single point and $K \cap D_2 = n$ points for some $n \geq 0$ as illustrated in Fig. 15(a), where D_1 and D_2 are meridian disks of V, each of which has a canceling disk in the complementary handlebody.

In Fig. 15(a), by cutting open V with the disks D_1, D_2, we can get a 3-ball B and $n+1$ strings in ∂B as in Fig. 15(b). Conversely, let $(B, t_1 \cup t_2 \cup \cdots \cup t_n \cup t_{n+1})$ be an $n+1$-string tangle such that $t_1 \cup \cdots \cup t_{n+1}$ is parallel to ∂B and by closing the tangle with $n+1$ strings we can get a (1, 1)-knot in ∂V for a standard genus two handlebody V, then we call $(B, t_1 \cup t_2 \cup \cdots \cup t_n \cup t_{n+1})$ a (1, 1)-tangle as in

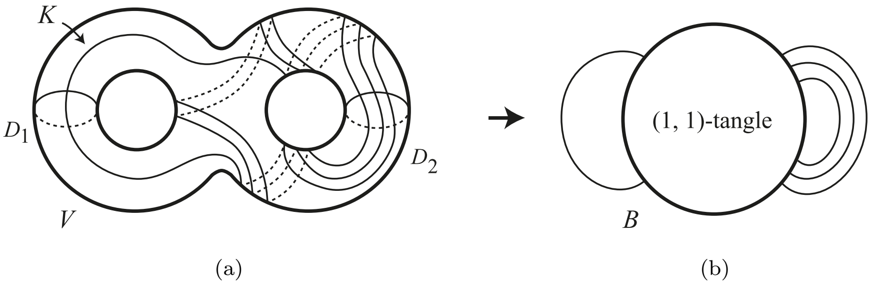


Fig. 15. (1, 1)-knot.

Fig. 15(b). We note that if $n = 1$ then a (1, 1)-tangle is a rational tangle. Then, we have:

Proposition 11. *Let K_1 be a 2-bridge knot and K_2 a $(1, 1)$-knot, and let $L = K_1 \cup K_2$ be a link illustrated in Fig.* 16(a). *Then, L is a tunnel number one link of type* II *not of type* I.

Proof. Let γ be an arc in the 3-ball B_1 which connects the two strings of the rational tangle so that γ is a level arc of the 2-string trivial tangle. Then, $K_1 \cup \gamma$ is deformed into a trivial glasses as in Fig. 16(b). This means that the complementary space of $K_1 \cup \gamma$ is a genus two handlebody, say V, and K_2 is in V. Then, since the (1, 1)-tangle in B_2 is parallel to ∂B_2 and since V can be regarded as a handlebody obtained by adding two 1-handles to B_2 as in Fig. 15, we see that K_2 is parallel to ∂V and there is a canceling meridian disk D intersecting K_2 in a single point. This shows that K_2 is a core of V and L is a tunnel number one link of type II. In addition, neither K_1 nor K_2 is a trivial knot. Then, by [3, Theorem 1.5] and Proposition 12(1) below, L is not of type I. This completes the proof. □

For essential tangle decompositions of tunnel number one links obtained in the above proposition, we have:

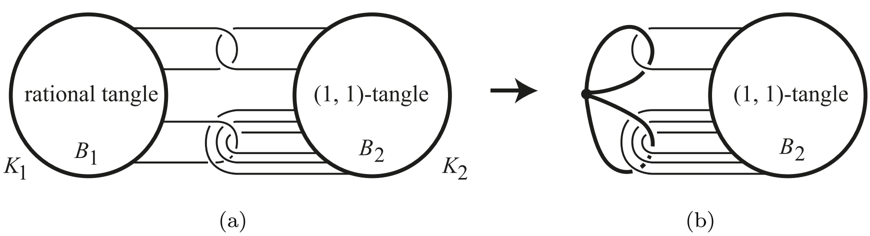


Fig. 16. $L = K_1 \cup K_2$.

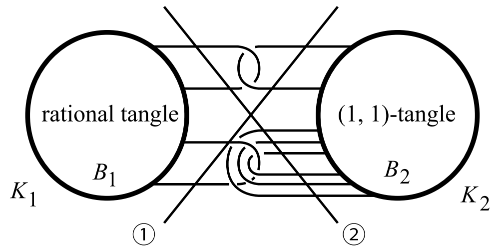


Fig. 17. Tangle decompositions.

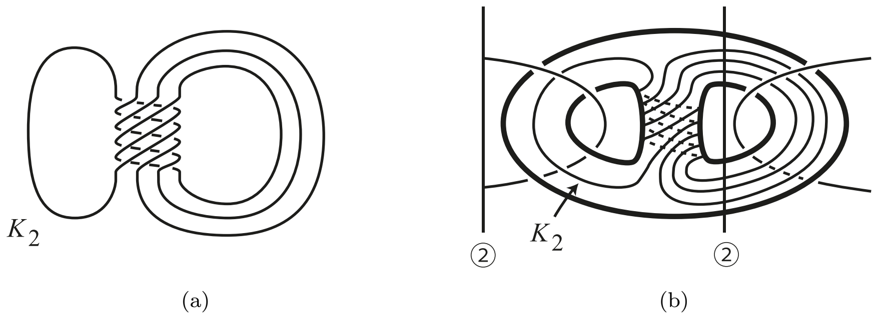


Fig. 18. $(n+1, n+2)$-torus knot.

Proposition 12. (1) *The link L in Proposition* 11 *has a* 2*-string essential tangle decomposition.* (2) *For any $n > 0$, there are infinitely many tunnel number one links of type* II *not of type* I *each of which has a* 2*-string essential tangle decomposition and an $n+1$-string essential tangle decomposition.*

Proof. (1) Decompose the link L in Proposition 11 into the two tangles with the line ① as in Fig. 17. Then the decomposition with the line ① is a 2-string essential tangle decomposition because K_1 and K_2 are non-trivial knots.

(2) For the links in Proposition 11, let K_2 be an $(n+1, n+2)$-torus knot for $n > 0$ as in Fig. 18(a). Decompose the link L into the two tangles with the line ② as in Fig. 17. Then we can see that the decomposition with the line ② is an $n+1$-string essential tangle decomposition as in Fig. 18(b). This completes the proof. □

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