Characterization of composite knots with 1-bridge genus two

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ABSTRACT

In the present paper, we characterize the knot types of composite knots in the 3-sphere S^3 with 1-bridge genus two.

1. Introduction

Let M be an orientable closed 3-manifold. Then it is well known that M can be decomposed into two handlebodies. We call the decomposition a Heegaard splitting of M and denote it by (V_1, V_2), i.e., $M = V_1 \cup V_2$, $V_1 \cap V_2 = \partial V_1 = \partial V_2$ and both V_1 and V_2 are handlebodies. Then the genus of V_1 (= the genus of V_2) is called the genus of the Heegaard splitting and the surface $\partial V_1 = \partial V_2$ is called the Heegaard surface of the Heegaard splitting.

Let K be a knot in an orientable closed 3-manifold M. Then the tunnel number $t(K)$ is the minimal number of mutually disjoint arcs in M such that each of the arcs has its end points in K and the exterior of the union of K and the arcs is a handlebody. This is equivalent to the minimal genus -1 among all Heegaard splittings (V_1, V_2) of M such that one of V_1 and V_2 contains K as a core of a handle. Next the 1-bridge genus $g_1(K)$ is the minimal genus among all Heegaard splittings (V_1, V_2) of M such that V_i intersects K in a single trivial arc in V_i for both $i = 1, 2$ (c.f. [2], [4] and [9]). Finally the h-genus $h(K)$ is the minimal genus among all Heegaard splittings (V_1, V_2) of M whose Heegaard surfaces contain K (c.f. [6]). Then by a little observation, we have :

Fact 1.1 *For any knot K in an orientable closed 3-manifold M,*

$$t(K) \leq g_1(K) \leq h(K) \leq t(K) + 1$$

The following examples show the difference among these three geometric invariants.

Example 1 Let K be a torus knot in S^3, then $t(K) = g_1(K) = h(K) = 1$.

Example 2 Let K be a 2-bridge knot in S^3, then $t(K) = g_1(K) = 1$ and $h(K) = 2$.

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Example 3 Let K be a knot in S^3 given in [9], then $t(K) = 1$ and $g_1(K) = h(K) = 2$.

Let K_1 and K_2 be knots in S^3. Then we denote the connected sum of K_1 and K_2 by $K_1 \# K_2$. Concerning the relationship between these geometric invariants and the connected sum, we get the following immediately.

Proposition 1.2 *Let K_1 and K_2 be knots in S^3, then*
(1) $t(K_1 \# K_2) \leq t(K_1) + t(K_2) + 1$,
(2) $g_1(K_1 \# K_2) \leq g_1(K_1) + g_1(K_2)$,
(3) $h(K_1 \# K_2) \leq h(K_1) + h(K_2)$.

On the lower bound of these invariants under the connected sum, the first result is :

Proposition 1.3 ([10, 11]) *Tunnel number one knots in S^3 are prime.*

This, by Fact 1.1, implies that 1-bridge genus one knots are prime and that h-genus one knots are prime.

Let B be a 3-ball and $t_1 \cup t_2$ be two arcs properly embedded in B, then $(B, t_1 \cup t_2)$ is called a 2-string tangle. We say that $(B, t_1 \cup t_2)$ is free if $cl(B - N(t_1 \cup t_2))$ is a genus two handlebody, where $N(t_1 \cup t_2)$ is a regular neighborhood of $t_1 \cup t_2$ in B, that $(B, t_1 \cup t_2)$ is essential if $cl(\partial B - N(t_1 \cup t_2))$ is incompressible in $cl(B - N(t_1 \cup t_2))$ and that t_i ($i = 1$ or 2) is unknotted if (B, t_i) is a trivial ball pair. Then we have shown the following.

Theorem 1.4 ([6]) *Let K_1 and K_2 be non-trivial knots in S^3. If $h(K_1 \# K_2) = 2$ then $h(K_1) = h(K_2) = 1$, i.e., both K_1 and K_2 are torus knots.*

Theorem 1.5 ([5, 7]) *Let K_1 and K_2 be non-trivial knots in S^3. If $t(K_1 \# K_2) = 2$, then one of the following holds.*

(1) $t(K_1) = t(K_2) = 1$ and $g_1(K_i) = 1$ for at least one of $i = 1, 2$ or

(2) K_1 or K_2, say K_1, is a 2-bridge knot, $t(K_2) = 2$ and K_2 satisfies the following condition $C(1)$.

$C(1)$: (S^3, K_2) is decomposed into two 2-string tangles such that both tangles are essential free tangles and at least one of the two tangles has an unknotted component.

In the present paper, we show the following :

Theorem 1.6 *Let K_1 and K_2 be non-trivial knots in S^3. If $g_1(K_1 \# K_2) = 2$, then one of the following holds.*
(1) $g_1(K_1) = g_1(K_2) = 1$,
(2) K_1 or K_2, say K_1, is a 2-bridge knot, $t(K_2) = 1$ and $g_1(K_2) = 2$ or

(3) *K_1 or K_2, say K_1, is a 2-bridge knot, $t(K_2) = 2$, $g_1(K_2) = 2$ and K_2 satisfies the following condition $C(2)$.*

$C(2)$: *(S^3, K_2) is decomposed into two 2-string tangles such that both tangles are essential free tangles and both tangles have an unknotted component.*

As a generalization of Theorem 1.6(2), we have the following, which will be proved at the end of the present paper :

Proposition 1.7 *Let K be a knot in S^3 with $g_1(K) = t(K)+1$. Then $g_1(K\#K') \leq g_1(K)$ for any 2-bridge knot K'.*

Now, let's cosider the knots K which satisfy the following condition $C(3)$, i.e., the complementary condition of $C(2)$ in $C(1)$.

$C(3)$: *(S^3, K) is decomposed into two 2-string tangles such that both tangles are essential free tangles, one of the two tangles has an unknotted component and the other has no unknotted component.*

Then by the above Theorems 1.5 and 1.6 we have $t(K) = 2$, $t(K\#K') = 2$ and $g_1(K\#K') = 3$ for any 2-bridge knot K'. Moreover we see that $g_1(K) = 2$ or 3. However we do not know if $g_1(K) = 2$ or 3 and this is a problem on the difference between tunnel number and 1-bridge genus. So we ask the following :

Problem *Determine the 1-bridge genus of knots K which satisfy the above condition $C(3)$.*

Throughout the present paper, we work in the piecewise linear category. For a manifold X and subcomplex Y in X, we denote a regular neighborhood of Y in X by $N(Y, X)$ or $N(Y)$ simply.

2. Preliminaries

Let V be a handlebody and γ an arc properly embedded in V. Let P be a surface (i.e. a connected 2-manifold) properly embedded in V with $P \cap \gamma = \emptyset$. Then we say that P is γ-inessential in V if P is compressible in $V - \gamma$ or is isotopic rel. ∂P to a surface in ∂V by an isotopy disjoint from γ, and that P is γ-essential if P is not γ-inessential.

Let K be a knot in S^3, and let (V_1, V_2) be a Heegaard splitting of S^3 which gives a 1-bridge decomposition of K, i.e., $V_i \cap K = \gamma_i$ is a trivial arc properly embedded in V_i for both $i = 1, 2$. Then we say that (V_1, V_2) is weakly K-reducible if there is a γ_i-essential disk D_i in V_i ($i = 1, 2$) such that $D_1 \cap D_2 = \emptyset$ and that (V_1, V_2) is strongly K-irreducible if it is not weakly K-reducible. The notion of weak reducibility and strong irreducibility of a Heegaard splitting is due to Casson and Gordon in [1], and some generalization related to 1-submanifolds have already been done by several people [2, 3, 4]. Let Q be a closed surface in S^3 intersecting K transversely, and let α be a simple closed curve in Q disjoint from K. Then we say that α is K-inessential if α bounds a disk in Q disjoint from K and that α is K-essential if α is not K-inessential. We say that Q is K-compressible if there is a disk, say D, in S^3 such that $D \cap K = \emptyset$, $D \cap Q = \partial D$ and ∂D is a K-essential simple closed curve, and that Q is K-incompressible if Q is not K-compressible.

Under the above notations, the following is due to Schultens and Hoiden (cf. an alternative proof due to the author).

Lemma 2.1 ([4, 8, 12]) *Let Q be a K-incompressible closed surface in $S^3 = V_1 \cup V_2$ intersecting K transversely. If the Heegaard splitting (V_1, V_2) is strongly K-irreducible, then Q can be isotoped rel. K so that Q intersects the Heegaard surface $\partial V_1 = \partial V_2$ in K-essential loops in both Q and the Heegaard surface.*

Now, let K_1 and K_2 be non-trivial knots in S^3, put $K = K_1 \# K_2$ and let S be the decomposing 2-sphere in S^3 giving the connected sum. Then S intersects K in two points, and any K-essential loop in S is a loop separating the two points $K \cap S$. Suppose K has 1-bridge genus two, i.e., S^3 has a genus two Heegaard splitting (V_1, V_2) such that K intersects V_i in a trivial arc, say γ_i, for both $i = 1, 2$. We divide the proof of Theorem 1.6 into the following two subcases.

3. Weakly K-reducible case

Suppose (V_1, V_2) is weakly K-reducible. Then there is an γ_i-essential disk D_i in V_i $(i = 1, 2)$ with $D_1 \cap D_2 = \emptyset$. The next lemma is a straightforward fact and we omit the proof.

Lemma 3.1 *Let V be a genus two handlebody and γ a trivial arc properly embedded in V, Let D be a γ-essential disk in V, then D is one of the following three types as in* Figure 1.

(i) *D is a non-separating disk in V,*

(ii) *D splits V into two solid tori or*

(iii) *D splits V into a genus two handlebody and a* 3*-ball containing γ.*

According to the types of disks D_1 and D_2, we have the following six cases.

Case (1) : Both D_1 and D_2 are of type (i).

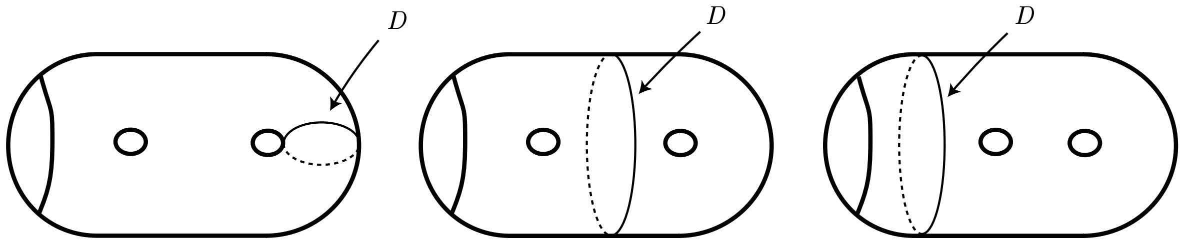

Figure 1

Let $N(D_i)$ be a regular neighborhood of D_i in V_i, and put $W_i = cl(V_i - N(D_i))$ $(i = 1, 2)$. Then $N(D_1)$ is a 2-handle for W_2 and $N(D_2)$ is a 2-handle for W_1. Let $N(\partial W_i)$ be a regular neighborhood of ∂W_i in W_i. Put $U_1^1 = cl(W_1 - N(\partial W_1))$, $U_2^2 = cl(W_2 - N(\partial W_2))$, $U_2^1 = N(\partial W_1) \cup N(D_2)$ and $U_1^2 = N(\partial W_2) \cup N(D_1)$. Then each of U_1^1 and U_2^2 is a solid torus, and each of U_2^1 and U_1^2 is (a solid torus $-$ a 3-ball) as in Figure 2. Then the intersection of $U_1^1 \cup U_2^1$ and $U_1^2 \cup U_2^1$ is a 2-sphere intersecting K in two points, and (U_1^i, U_2^i) extends to a genus one Heegaard splitting of a 3-sphere which gives a 1-bridge decomposition of the knots K_1' and K_2' with $K_1' \# K_2' = K_1 \# K_2$. Then the uniqueness of the prime decomposition of knots, we have $g_1(K_1) = g_1(K_2) = 1$.

Case (2) : D_1 is of type (i) and D_2 is of type (ii).

In this case, we can find a γ_2-essential disk of type (i) in V_2, say D_2', with $D_1 \cap D_2' = \emptyset$. Hence this case is the same as Case (i) and we have $g_1(K_1) = g_1(K_2) = 1$.

Case (3) : D_1 is of type (i) and D_2 is of type (iii).

Let $N(D_1)$ be a regular neighborhood of D_1 in V_1 and put $W_1 = cl(V_1 - N(D_1))$. Since D_2 is a separating disk in V_2, D_2 splits V_2 into two pieces, say W_2^1 and W_2^2, where W_2^1 contains γ_2. Then W_2^1 is attached to W_1 and $N(D_1)$ is attached to W_2^2. Let $N(\partial W_1)$ be a regular neighborhood of ∂W_1 in W_1 and $N(\partial W_2^2)$ be a regular neighborhood of ∂W_2^2 in W_2^2. Put $U_1^1 = cl(W_1 - N(\partial W_1))$, $U_2^2 = cl(W_2^2 - N(\partial W_2^2))$, $U_2^1 = N(\partial W_1) \cup W_2^1$ and $U_1^2 = N(\partial W_1^2) \cup N(D_1)$. Then (U_1^1, U_2^1) is a genus one Heegaard splitting, (U_1^2, U_2^2) is a genus two Heegaard splitting and the intersection of $U_1^1 \cup U_2^1$ and $U_1^2 \cup U_2^2$ is a torus $T = U_2^1 \cap U_1^2$ as in Figure 3. We note that T is incompressible in $S^3 - K$.

If (U_1^1, U_2^1) is weakly K-reducible, then we see that K is a trivial knot. If (U_1^2, U_2^2) is weakly reducible, then we see that $U_1^2 \cup U_2^2$ is a solid torus and K has 1-bridge genus one. These contradictions show that (U_1^1, U_2^1) is strongly K-irreducible and (U_1^2, U_2^2) is

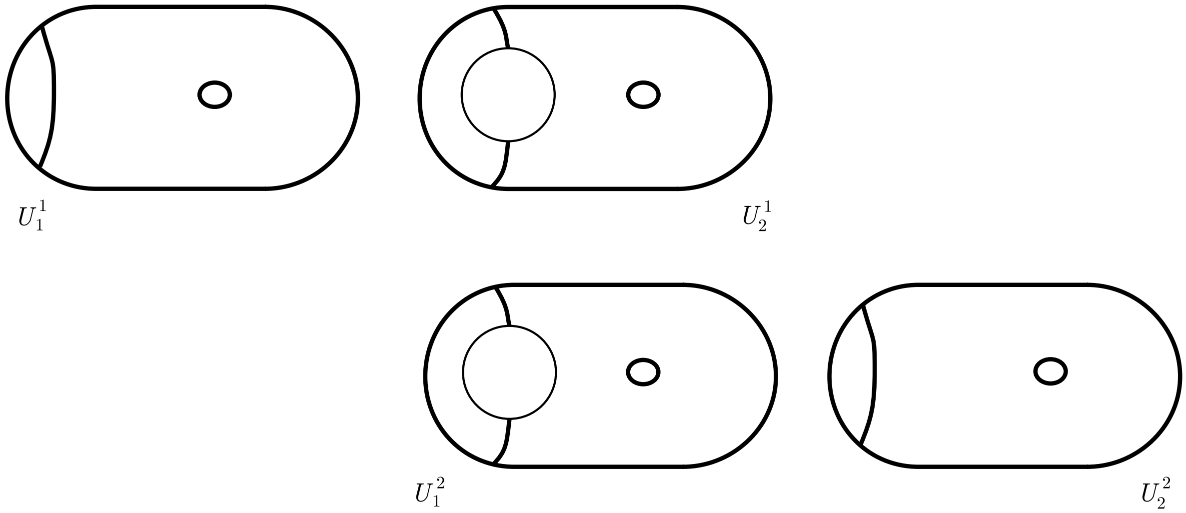

Figure 2

strongly irreducible. Put $F_1 = \partial U_1^1$ and $F_2 = \partial U_2^2$. Then by Lemma 2.1 and the fact that T is incompressible in $S^3 - K$, S can be isotoped rel. K so that each component of $S \cap (F_1 \cup F_2 \cup T)$ is K-essential in S. Let D_1^* and D_2^* be the closure of the two open disk components of $S - (F_1 \cup F_2 \cup T)$, then, since any disk properly embedded in U_2^1 intersecting K in a single point is isotopic rel. K to a disk in the boundary, both D_1^* and D_2^* are meridian disks in U_1^1 parallel to each other. If there is an annulus component of $S-(F_1 \cup F_2 \cup T)$ in U_1^1, say A, then A is a compressing annulus in U_1^1 and the compressing disk, say E, intersects K in a single point. Then by a standard cut and paste operation along E, we get a 2-sphere S' giving a connected sum of K with $|S' \cap F| < |S \cap F|$, a contradiction. Hence $U_1^1 \cap S = D_1^* \cup D_2^*$ as in Figure 4.

Since $F_1 \cap S = \partial(D_1^* \cup D_2^*)$ and any incompressible annulus properly embedded in U_2^1 with the boundary in T is isotopic to an annulus in T, we have the following two subcases : (i) $S \cap U_2^1 = A$ a separating annulus if $S \cap T = \emptyset$ or (ii) $S \cap U_2^1 = A_1 \cup A_2$ two annuli connecting F_1 and T if $S \cap T \neq \emptyset$ as in Figure 4.

Suppse we are in case (i). Let X_1 and X_2 be the closure of each component of $U_1^1 - (D_1^* \cup D_2^*)$ and let Y_1 and Y_2 be the closure of each component of $U_2^1 - A$, where X_1 contains the two points of $\partial(U_1^1 \cap K)$ and Y_1 contains $U_2^1 \cap K$. Then $X_1 \cap \partial U_1^1$ is identified with $Y_1 \cap \partial U_2^1$. Let D be a disk and x a point in Int$D$ and put $\delta = \{x\} \times I$, where $I = [0, 1]$. Then δ is a trivial arc in the 3-ball $D \times I$. Put $B_1 = X_1$ and $B_2 = Y_1 \cup (D \times I)$, where $A \subset \partial Y_1$ is identified with $\partial D \times I$. Then both B_1 and B_2 are 3-balls, $B_1 \cup B_2$ is a

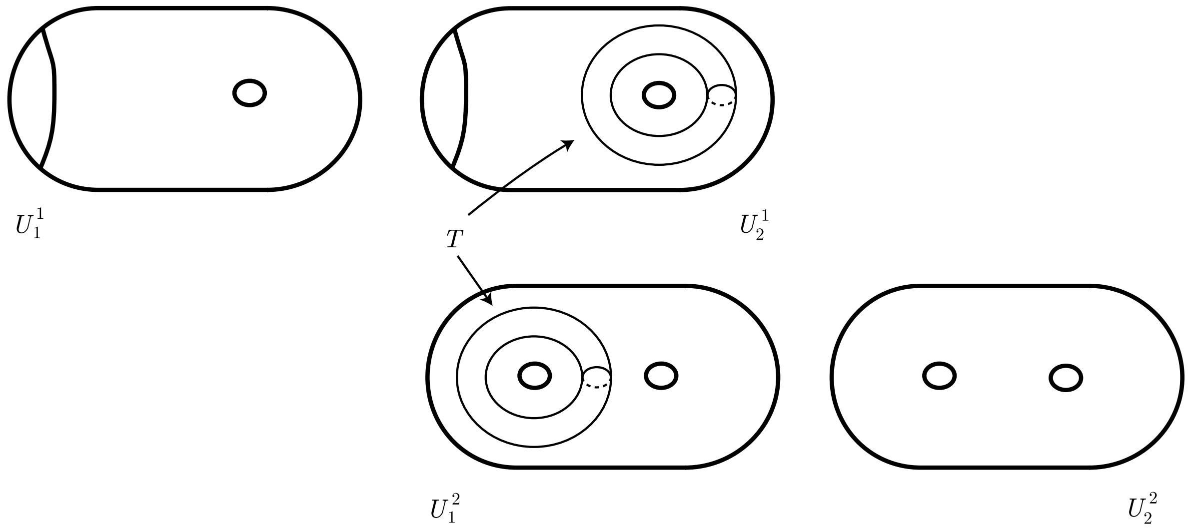

Figure 3

3-sphere and (B_1, B_2) gives a 2-bridge decomposition of the knot $(X_1 \cap K) \cup (Y_1 \cap K) \cup \delta$. By changing the letters if necessary, we may assume that the knot in the 3-sphere $B_1 \cup B_2$ is the knot K_1. Hence K_1 is a 2-bridge knot.

On the other hand, let D' be a disk, y a point in $\mathrm{Int}D'$ and put $\delta' = \{y\} \times I$. Put $Z = X_2 \cup (D' \times I)$, where $D_1^* \cup D_2^*$ is identified with $D' \times \{0, 1\}$. Then Z is a solid torus and the knot $(X_2 \cap K) \cup \delta'$ is a core of the solid torus. Since Y_2 is a (torus $\times I$), $Z \cup Y_2$ is a solid torus too, where $\partial D' \times I$ is identified with $A \subset \partial Y_2$. Put $U_0 = (Z \cup Y_2) \cup_T U_1^2$. Then U_0 is a genus two handlebody and (U_0, U_2^2) is a genus two Heegaard splitting of a 3-sphere. Moreover, the knot $(X_2 \cap K) \cup \delta'$ $(= K_2)$ is a core of a handle of U_0. This shows that K_2 has tunnel number one.

Suppose we are in case (ii). Put $A_3 = cl(S - (D_1^* \cup D_2^* \cup A_1 \cup A_2))$. Then A_3 is an incompressible annulus properly embedded in $U_1^2 \cup U_2^2$. If A_3 is in U_1^2, then A_3 is isotopic to an annulus in T and this case is reduced to case (i).

Hence we can put $A_3 = (A' \cup A'' \cup B_1 \cup \cdots \cup B_\ell) \cup (C_1 \cup \cdots \cup C_{\ell+1})$, where $(A' \cup A'' \cup B_1 \cup \cdots \cup B_\ell) \subset U_1^2$, $(C_1 \cup \cdots \cup C_{\ell+1}) \subset U_2^2$ and $\partial B_i \subset F_2$ $(i = 1, 2, \cdots, \ell)$. If some C_i is a separating annulus, then we can put $C_i = D_i \cup b_i$, where D_i is a separating essential disk and b_i is a band. Then b_i is contained in a solid torus component cut off by D_i, and b_i winds around a longitude of the solid torus with p times for some $p > 0$. Let G_1 and G_2 be the closure of the two components of $S - C_i$. Then we can regard G_1 as a 2-handle for the solid torus. Hence we have $p = 1$ and C_i is isotopic to an annulus

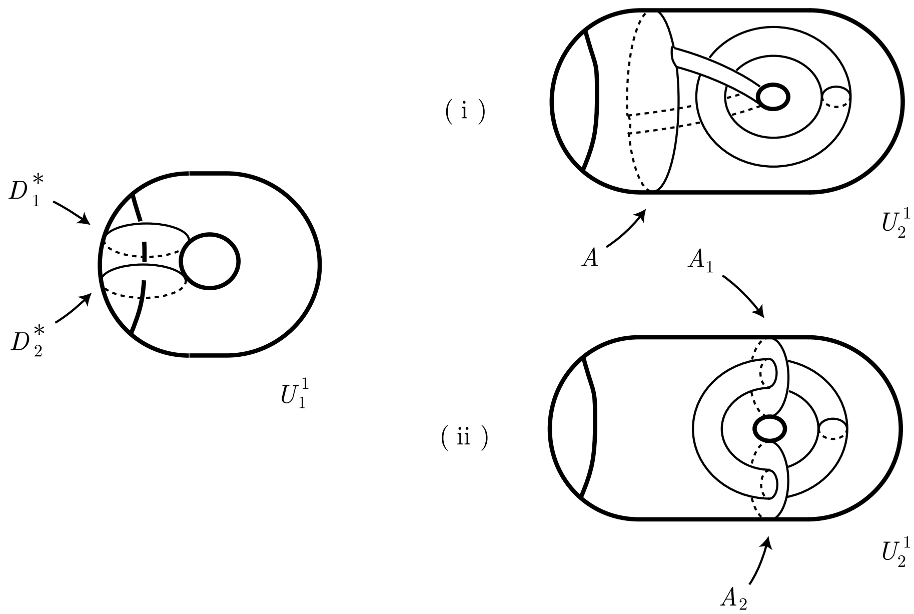

Figure 4

in ∂U_2^2. Thus we see that C_i is a non-separating annulus for any i $(1 \leq i \leq \ell)$. Then we can put $C_i = D_i \cup b_i$, where D_i is a non-separating disk in U_2^2 and b_i is a band. Moreover by the same reason as above, b_i winds around a longitude of the solid torus cut off by C_i exactly once. This means that $C_1, C_2, \cdots, C_{\ell+1}$ are all mutually parallel annuli, and $\partial(C_1 \cup \cdots \cup C_{\ell+1})$ consists of two parallel classes in ∂U_2^2 each of which has $\ell + 1$ loops.

Next, suppose some B_i is a separating annulus in U_1^2, and put $B_i = D_i \cup b_i$, where D_i is a separating essential disk in U_1^2 and b_i is a band. If b_i is contained in the solid torus component cut off by D_i, then we have a contradiction as above. Hence b_i is contained in the (torus $\times I$) component cut off by D_i. Then this means that ∂B_i, $A' \cap F_2$, $A'' \cap F_2$ are all mutually parallel loops in F_2. Moreover, if there is a non-separating annulus, say B_j, one of the two components of ∂B_j is parallel to a component of ∂B_i and the other is not. Hence $(A' \cap F_2) \cup (A'' \cap F_2) \cup \partial(B_1 \cup \cdots \cup B_\ell)$ cannot be divided into two parallel classes each of which has $\ell + 1$ loops.

After all, we see that $B_1, \cdots, B_\ell$ are all non-separating annuli, and A' and A'' are not mutually parallel annuli because $(A' \cap F_2) \cup (A'' \cap F_2) \cup \partial(B_1 \cup \cdots \cup B_\ell)$ consists of two parallel classes each of which has $\ell + 1$ loops. Put $a' = A' \cap F_2$ and $a'' = A'' \cap F_2$. Then a' and a'' are two components of $\partial(C_1 \cup \cdots \cup C_{\ell+1})$, which are not mutually parallel. Then we can find a non-separating annulus A^* in U_2^2 with $\partial A^* = a' \cup a''$. Thus

$A^* \cup A' \cup A'' \cup A_1 \cup A_2 \cup D_1^* \cup D_2^*$ is a non-separating 2-sphere in the 3-sphere containing K. This contradiction shows that the case (ii) does not occur, and completes the proof of Case (3).

Case (4) : Both D_1 and D_2 are of type (ii). In this case, we can find a non-separating disk D_1' in V_1 with $D_1' \cup D_2 = \emptyset$. Hence by Case (2) we have $g_1(K_1) = g_1(K_2) = 1$.

Case (5) : D_1 is of type (ii) and D_2 is of type (iii). In this case, we can find a non-separating disk D_1' in V_1 with $D_1' \cup D_2 = \emptyset$. Hence by Case (3) we have one of K_1 and K_2, say K_1, is a 2-bridge knot and K_2 has tunnel number one.

Case (6) : Both D_1 and D_2 are of type (iii). Let E_i be the closure of the component of $\partial V_i - \partial D_i$ with $\partial \gamma_i \subset E_i$ $(i = 1, 2)$. Then by $D_1 \cap D_2 = \emptyset$ and by changing the letters of D_1 and D_2 if necessary, we may assume that E_1 is contained in E_2. Then, since ∂E_1 is parallel to ∂E_2 in E_2, we may assume that $\partial D_1 = \partial D_2$. This shows that K is a trivial knot, a contradiction.

After all, we see that if (V_1, V_2) is weakly K-reducible, then $g_1(K_1) = g_1(K_2) = 1$ or one of K_1 and K_2, say K_1, is a 2-bridge knot and $t(K_2) = 1$. In the latter conclusion, if $g_1(K_2) = 1$, then $g_1(K_1) = g_1(K_2) = 1$, and if $g_1(K_2) = 2$, then $t(K_2) = 1$ and $g_1(K_2) = 2$. Hence we get the conclusion (1) or (2) in our theorem, and complete the proof of the weakly K-reducible case $\square$.

4. Strongly K-irreducible case

Suppose (V_1, V_2) is strongly K-irreducible, and put $F = \partial V_1 = \partial V_2$. Then by Lemma 2.1, S is isotoped rel. K so that each component of $F \cap S$ is a K-essential loop in both S and F. Then the closure of the components of $S - F$ consists of two disks and several annuli, where each disk intersects K in a single point and each annulus is a γ_1 or γ_2-essential annulus in V_1 or in V_2 respectively. We assume that $|S \cap F|$ is minimal among all 2-spheres that give the connected sum of $K = K_1 \# K_2$ and that intersect F in K-essential loops in both S and F.

Lemma 4.1 *Let A be the closure of an open annulus component of $S - F$ and suppose there is a solid torus V in S^3 such that $S \cap V = S \cap \partial V = A$ and A is incompressible in V. Then A winds around a longitude of V exactly once.*

Proof. Since A is incompressible in V, A winds around a longitude of V with p times for some $p > 0$. Let G be a closure of a component of $S - A$, then G is a disk and $G \cap V = G \cap \partial V = \partial G$ is a loop which winds around a longitude of V with p times. This means that S^3 contains a lens space of the order p. Hence $p = 1$ and this completes the proof of the lemma □.

Let D_1^* and D_2^* be the closure of the disk components of $S - F$. Then by changing the letters if necessary, we have the following two cases, Case I : $D_1^* \subset V_1$ and $D_2^* \subset V_2$, Case II : $D_1^* \cup D_2^* \subset V_1$.

Suppose we are in Case I. Then we can put $V_1 \cap S = D_1^* \cup A_1 \cup \cdots \cup A_\ell$ and $V_2 \cap S = D_2^* \cup B_1 \cup \cdots \cup B_\ell$, where A_i (B_i resp.) is a γ_1-essential annulus in V_1 (γ_2-essential annulus in V_2 resp.) ($i = 1, 2, \cdots, \ell$).

Case I-(1) : D_1^* is a separating disk in V_1. Since D_1^* intersects γ_1 in a single point, D_1^* splits V_1 into two solid tori. Then A_1 is a γ_1-essential annulus in one of the two solid tori. If A_1 is a non-separating annulus, then A_1 is a compressing annulus in V_1 and the compressing disk, say E_1, intersect γ_1 in a single point. Then by a standard cut and paste operation along E_1, we get a 2-sphere S' giving a connected sum of K with $|S' \cap F| < |S \cap F|$, a contradiction. If A is a separating annulus, then by the argument as above and by Lemma 4.1, A winds around a handle of V_1 exactly once, and we can push out the annulus from V_1 by an isotopy. This contradicts the minimality of $|S \cap F|$ again. Hence we see that $V_1 \cap S = D_1^*$, $V_2 \cap S = D_2^*$ and $S = D_1^* \cup D_2^*$. Then D_1^* splits V_1 into two solid tori U_1 and U_2, D_2^* splits V_2 into two solid tori W_1 and W_2 and we may assume that $U_1 \cap \partial V_1$ ($U_2 \cap \partial V_1$ resp.) is identified with $W_1 \cap \partial V_2$ ($W_2 \cap \partial V_2$ resp.). Then both (U_1, W_1) and (U_2, W_2) extend to 1-bridge decompositions of K_1 and K_2 respectively, and we have $g_1(K_1) = g_1(K_2) = 1$.

Case I-(2) : D_1^* is a non-separating disk in V_1. In this case, by the proof of Case I-(1), D_2^* is a non-separating disk too. Suppose some A_i is a separating annulus in V_1. Then we can put $A_i = D_i \cup b_i$, where D_i is a γ_1-essential disk in V_1 and b_i is a band. Since $D_1^* \cap D_i = \emptyset$, D_i splits V_1 into two solid tori U_1 and U_2 with $\gamma_1 \subset U_1$. If b_i is contained in U_2, then by Lemma 4.1, b_i winds around U_2 exactly once and we can push out A_i from V_1. This contradicts the minimality of $|S \cap F|$. Hence b_i is contained in U_1.

Since D_1^* is a meridian disk of U_1, A_i is a compressing annulus in U_1. If a component of ∂A_i bounds a disk in ∂U_1, then the intersection of the disk and $\partial \gamma_1$ consists of 0, 1 or 2 points. If it is 0 or 2 points, then we have a contradiction because the linking number of a meridian of a knot and the knot is ± 1. If it is 1 point, then by a standard cut and paste operation along the disk we get a 2-sphere S' giving a connected sum of K with $|S' \cap F| < |S \cap F|$, a contradiction. Hence each component of ∂A_i is a meridian of U_1.

Let a_1 and a_2 be the two components of ∂A_i and let E_1 and E_2 be two annuli in ∂U_1 with $\partial E_1 = \partial D_1^* \cup a_1$, $\partial E_2 = \partial D_1^* \cup a_2$ and $E_1 \cap E_2 = \partial D_1^*$. Then by changing the letters if necessary, we may assume that $E_1 \cap \partial \gamma_1 = 0$ or 1 point. Put $E_0 = D_1^* \cup E_1$ and let E_0'

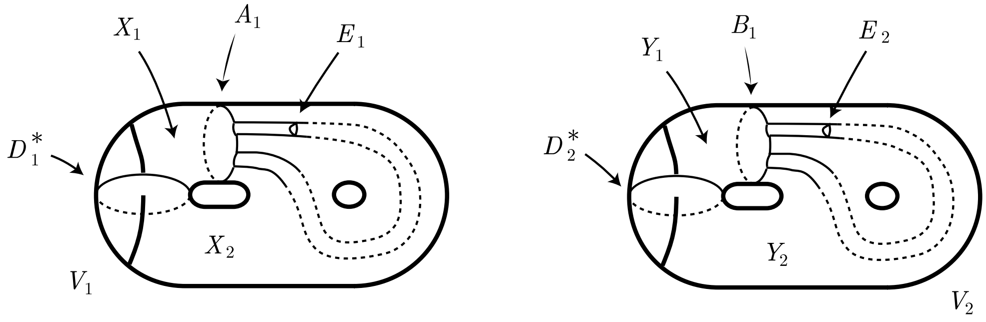

Figure 5

be the disk obtained by slightly pushing E_0 off with $E_0' \cap D_1^* = \emptyset$. Then $E_0' \cap A_i = a_1$ and $E_0' \cap \gamma_1 = 1$ or 2 points. Then we have a contradiction as above.

Therefore $A_1, A_2, \cdots, A_\ell$ are all non-separating annuli in V_1, and we can put $A_i = D_i \cup b_i$, where D_i is a non-separating disk and b_i is a band in V_1 ($i = 1, 2, \cdots, \ell$). If $\{D_1^*, D_i\}$ is a complete meridian disk system of V_1 for some i, then, since $D_1^* \cup D_i$ cuts V_1 open into a 3-ball, A_i is a compressing annulus. Then we have a contradiction as above. Hence each D_i is parallel to D_1^* and b_i is contained in the solid torus obtained by cut open along $D_1^* \cup D_i$. Then, by Lemma 4.1, A_i winds around a longitude of the solid torus exactly once. Moreover two such annuli in a solid torus are mutually parallel, we see that $A_1, A_2, \cdots, A_\ell$ are all mutually parallel annuli in V_1. By the same argument as these, we see that $B_1, B_2, \cdots, B_\ell$ are mutually parallel annuli in V_2. Then, since the region in ∂V_1 bounded by $\partial(D_1^* \cup A_1)$ is identified with the region in ∂V_2 bounded by $\partial(D_2^* \cup B_1)$ or by $\partial(D_2^* \cup B_\ell)$, we have $\ell = 1$ as in Figure 5.

Let E_1 (E_2 resp.) be a boundary compressing disk for A_1 (B_1 resp.) in V_1 (V_2 resp.), and X_1 and X_2 (Y_1 and Y_2 resp.) be the closure of the components of $V_1 - (D_1^* \cup A_1)$ ($V_2 - (D_2^* \cup B_1)$ resp.) with $E_1 \subset X_1$ ($E_2 \subset Y_1$ resp.) as in Figure 5. If $X_1 \cap \partial V_1$ is identified with $Y_2 \cap \partial V_2$, then $E_1 \cap E_2 = \emptyset$. Perform a boundary compression for B_1 along E_2 and let b be the band in V_1 produced by the compression. Then, by $E_1 \cap E_2 = \emptyset$, we see that $b \cap E_1 = \emptyset$, and hence we can performe a boudary compression for A_1 along E_1 leaving b in V_1. This shows that S is isotopic rel. K to a 2-sphere S' with $|S' \cap F| < |S \cap F|$, a contradiction. Thus $X_i \cap \partial V_1$ is identified with $Y_i \cap \partial V_2$ ($i = 1, 2$).

Put $\alpha_1 = X_1 \cap K$ and $\alpha_2 = Y_1 \cap K$. Let D, D' be two disks, x (x' resp.) a point in $\mathrm{Int} D$ ($\mathrm{Int} D'$ resp.) and put $\beta_1 = x \times [0,1]$ in $D \times [0,1]$ and $\beta_2 = x' \times [0,1]$ in $D' \times [0,1]$. Put $G_1 = X_1 \cup_{A_1 = \partial D \times [0,1]} D \times [0,1]$ and $G_2 = Y_1 \cup_{B_1 = \partial D' \times [0,1]} D' \times [0,1]$. Then both $(G_1, \alpha_1 \cup \beta_1)$

and $(G_2, \alpha_2 \cup \beta_2)$ are 2-string trivial tangles. Hence $(G_1, \alpha_1 \cup \beta_1) \cup (G_2, \alpha_2 \cup \beta_2)$ gives a 2-bridge decomposition of a knot in $S^3 = G_1 \cup G_2$, and we may assume that the knot is K_1 by changing the letters of K_1 and K_2 if necessary

Next put $\varepsilon_1 = X_2 \cap K$, $\varepsilon_2 = Y_2 \cap K$. Let E, E' be two disks and y (y' resp.) a point in Int$E$ (IntE' resp.), and put $\delta_1 = y \times [0,1]$ in $E \times [0,1]$ and $\delta_2 = y' \times [0,1]$ in $E' \times [0,1]$. Put $P_1 = X_2 \cup_{A_1 = \partial E \times [0,1]} E \times [0,1]$ and $P_2 = Y_2 \cup_{B_1 = \partial E' \times [0,1]} E' \times [0,1]$. Then, since A_1 (B_1 resp.) winds around a longitude of the solid torus X_2 (Y_2 resp.) exactly once by Lemma 4.1, both P_1 and P_2 are 3-balls, and both $(P_1, \varepsilon_1 \cup \delta_1)$ and $(P_2, \varepsilon_2 \cup \delta_2)$ are 2-string tangles and the knot $\varepsilon_1 \cup \varepsilon_2 \cup \delta_1 \cup \delta_2$ is K_2 in the 3-sphere $P_1 \cup P_2$. If one of the two tangles is inessential, say $(P_1, \varepsilon_1 \cup \delta_1)$, then there is a compressing disk for $\partial P_1 - (\varepsilon_1 \cup \delta_1)$ in $P_1 - (\varepsilon_1 \cup \delta_1)$ and the disk separates ε_1 from δ_1. Then, since $cl(P_1 - N(\varepsilon_1 \cup \delta_1))$ is a genus two handlebody, $(P_1, \varepsilon_1 \cup \delta_1)$ is a trivial tangle. This means that there is a boundary compressing disk E'_1 for A_1 in V_1 with $E'_1 \subset X_2$. Then for a boundary compressing disk E_2 for B_1 in V_2 with $E_2 \subset Y_1$, we have $E'_1 \cap E_2 = \emptyset$. Then by the above argument, S is isotopic rel. K to a 2-sphere S' with $|S' \cap F| < |S \cap F|$, a contradiction. Hence $(P_1, \varepsilon_1 \cup \delta_1)$ is an essential tangle, and so is $(P_2, \varepsilon_2 \cup \delta_2)$. Since $cl(P_1 - N(\varepsilon_1 \cup \delta_1)) = cl(X_2 - N(\varepsilon_1))$ is a genus two handlebody, $(P_1, \varepsilon_1 \cup \delta_1)$ is a free tangle and so is $(P_2, \varepsilon_2 \cup \delta_2)$. Moreover, since $cl(P_1 - N(\delta_2)) = X_2$ is a solid torus, δ_1 is a trivial arc in P_1 and so is δ_2 in P_2. Hence the knot K_2 has a tangle decomposition satisfying the condition C(2).

In this case we have $t(K_2) \geq 2$ because tunnel number one knots have no 2-string essential tangle decomposition by [11]. Put $P'_2 = cl(P_2 - N(\delta_2))$ and put $P'_1 = P_1 \cup N(\delta_2)$. Then P'_2 ($= Y_2$) is a solid torus and ε_2 is a trivial arc in P'_2, and $\varepsilon_1 \cup \delta_2 \cup \delta_1$ is an arc in P'_1. Let δ'_1 be an arc properly embedded in P_1 parallel to δ_1, and let $N(\delta'_1)$ be a regular neighborhood of δ'_1 in P_1 with $N(\delta'_1) \cap \delta_1 = \emptyset$ and $N(\delta'_1) \cap N(\delta_2) = \emptyset$. Put $Q_1 = cl(P'_1 - N(\delta'_1))$ and $Q_2 = P'_2 \cup N(\delta'_1)$. Then both Q_1 and Q_2 are genus two handlebodies and, since $\varepsilon_1 \cup \delta_1$ is a 2-string trivial arc system in the solid torus $cl(P_1 - N(\delta'_1))$, $\varepsilon_1 \cup \delta_2 \cup \delta_1$ is a trivial arc in Q_1 and so is ε_2 in Q_2. Hence the knot $K_2 = \varepsilon_1 \cup \delta_2 \cup \delta_1 \cup \varepsilon_2$ has 1-bridge genus two. Thus we have $2 \leq t(K_2) \leq g_1(K_2) \leq 2$, and this implies $t(K_2) = g_1(K_2) = 2$. Therefore we get the conclusion (3) in our theorem.

Suppose we are in Case II, i.e. $D^*_1 \cup D^*_2 \subset V_1$.

Case II-(1) : One of D^*_1 and D^*_2, say D^*_1, is a separating disk. By the argument in Case I-(1), $S \cap V_1 = D^*_1 \cup D^*_2$ and D^*_2 is a separating disk parallel to D^*_1. Then $S \cap V_2 = B_1$ is a separating γ_1-essential annulus in V_2.

Lemma 4.2 *Let B be the closure of an open annulus component of $S - F$ in V_2. If B is a separating incompressible annulus in V_2, then B is ∂-parallel and the parallelism*

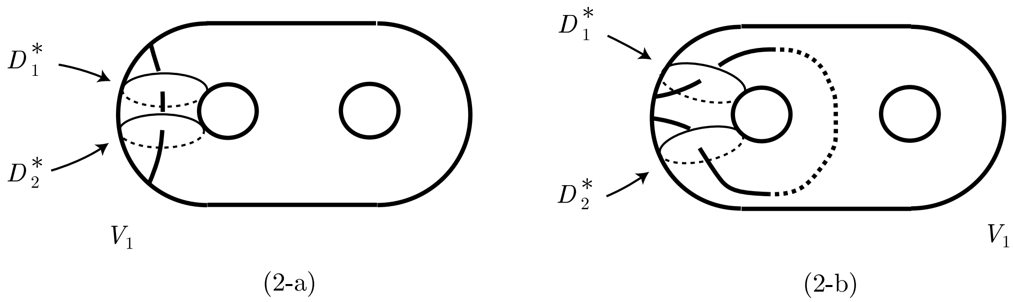

Figure 6

contains γ_2.

Proof. Since B is γ_2-essential, we can put $B = D \cup b$, where D is a γ_2-essential separating disk in V_2 and b is a band. If D cuts off a 3-ball containing γ_2, then the conclusion holds. Suppose D splits V_2 into two solid tori W_1 and W_2 with $\gamma_2 \subset W_1$. If b is contained in W_2, then by considering an inner annulus of $S - F$ parallel to B if necessary and by Lemma 4.1, B winds around a longitude of W_2 exactly once. Then we can push out B from V_2 into V_1 by an isotopy. This contradicts the minimality of $|S \cap F|$. If b is contained in W_1, then by the same reason as above B winds around a longitude of W_1 exactly once. Hence B is ∂-paralle and the parallelism contains γ_2. This completes the proof of the lemma $\square$.

By this lemma, if B_1 is incompressible in V_2, then the annulus region in ∂V_2 bounded by ∂B_1 contains $\partial\gamma_2$. However, the annulus region in ∂V_1 bounded by $\partial(D_1^* \cup D_2^*)$ contains no point of $\partial\gamma_1$, a contradiction. Suppose B_1 is compressible in V_2. Since B_1 is a γ_2-essential annulus in V_2, $B_1 = D_1 \cup b_1$, where D_1 is a γ_2-essential disk and b_1 is a band. If D_1 cuts off a 3-ball in V_2, then we have a contradiction as above. Suppose D_1 splits V_2 into two solid tori. Then, since the annulus region in ∂V_2 bounded by ∂B_1 contains no point of $\partial\gamma_2$, we have a contradiction which shows that Case II-(1) does not occur. Hence hereafter both D_1^* and D_2^* are non-separating disks in V_1.

Case II-(2) : D_1^* and D_2^* are mutually parallel non-separating disks in V_1. Let C be the annulus region bounded by $\partial(D_1^* \cup D_2^*)$ in ∂V_1. Then we have the following two subcases as in Figure 6, case (2-a) : $\partial\gamma_1 \not\subset C$, case (2-b) : $\partial\gamma_1 \subset C$.

Suppose we are in case (2-a). If $S \cap V_1 = D_1^* \cup D_2^*$, then $S \cap V_2 = B_1$ is a separating annulus and we have a contradiction as in Case II-(1). Thus we can put $V_1 = D_1^* \cup D_2^* \cup A_1 \cup \cdots \cup A_\ell$. Then by the argument of Case I-(2), $A_1 \cup A_2 \cup \cdots \cup A_\ell$ are all mutually parallel

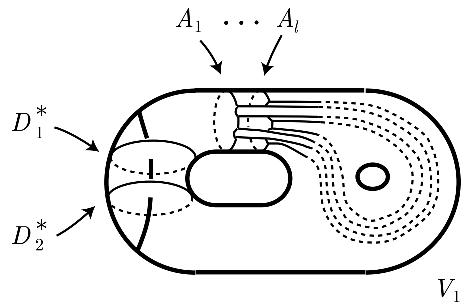

Figure 7

non-separating annuli, such that $A_i = D_i \cup b_i$ and D_i is parallel to D_1^* $(i = 1, 2, \cdots, \ell)$ as illustrated in Figure 7.

If $S \cap V_2 = B_1 \cup B_2 \cup \cdots \cup B_{\ell+1}$ contains a separating annulus, then we have a contradiction by the argument similar to the proof of Case II-(1). Hence $B_1 \cup B_2 \cup \cdots \cup B_{\ell+1}$ are all non-separating annuli in V_2. Since we can find a simple closed curve (a core of a handle of V_1) which intersects each component of $D_1^* \cup D_2^* \cup A_1 \cup \cdots \cup A_\ell$ in a single point, and since S is a separating 2-sphere in S^3, we see that ℓ is even.

Put $B_i = E_i \cup c_i$, where E_i is a non-separating disk in V_2 and c_i is a band $(i = 1, 2, \cdots, \ell + 1)$. If $E_1 \cup E_2 \cup \cdots \cup E_{\ell+1}$ consists of one or two parallel classes, then we can find a simple closed curve which intersects each component of $E_1 \cup E_2 \cup \cdots \cup E_{\ell+1}$ in a single point. Then since $\ell + 1$ is odd, the intersection number of the loop and S is odd, a contradiction. Hence $E_1 \cup E_2 \cup \cdots \cup E_{\ell+1}$ consists of three parallel classes $E_1 \cup \cdots \cup E_j$, $E_{j+1} \cup \cdots \cup E_k$ and $E_{k+1} \cup \cdots \cup E_{\ell+1}$. Then, since each component of $\partial(D_1^* \cup D_2^* \cup A_1 \cup \cdots \cup A_\ell)$ is an essential loop in ∂V_1, we may assume that the bands $c_{j+1} \cup \cdots \cup c_k$ and $c_{k+1} \cup \cdots \cup c_{\ell+1}$ run over the bands $c_1 \cup \cdots \cup c_j$ as in Figure 8.

Let G_1 $(G_2$ resp.) be a non-separating disk in V_2 parallel to E_{j+1} $(E_{k+1}$ resp.) with $G_1 \cap S = \emptyset$ $(G_2 \cap S = \emptyset$ resp.). Then $G_1 \cup G_2$ is a complete merdian disk system of V_2, and $(G_1 \cup G_2) \cap (D_1^* \cup D_2^*) = \emptyset$. This means that $H_1(S^3)$ is a non-trivial group. This contradiction shows that Case II-(2-a) does not occur.

Suppose we are in case (2-b). Put $S \cap V_1 = D_1^* \cup D_2^* \cup A_1 \cup \cdots \cup A_\ell$. Suppose $A_1 \cup \cdots \cup A_\ell \neq \emptyset$. Recall $A_1 = D_1 \cup b_1$. If D_1 is a non-separating disk, then since $D_1^* \cup D_1$ is a complete meridian disk system of V_1, A_1 is a compressing annulus. If D_1 is a

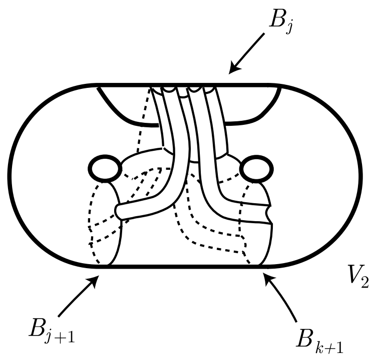

Figure 8

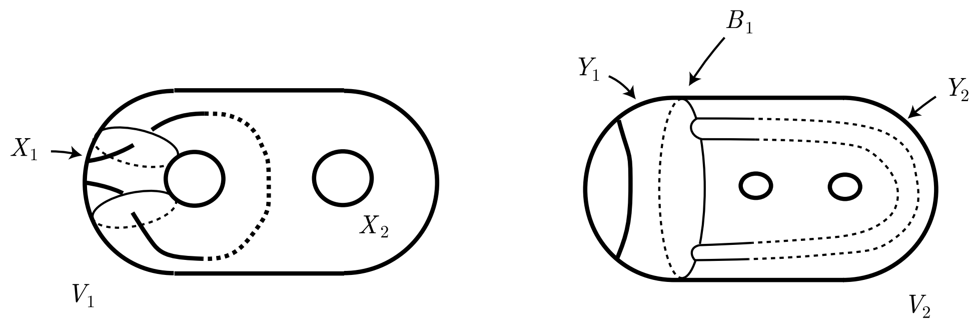

Figure 9

separating disk, then by Lemma 4.1, A_1 is a compressing annulus. Hence, in both cases, A_1 is a compressing annulus in V_1. Then by the argument similar to the proof of Case I-(2), we have a contradiction. Hence $S \cap V_1 = D_1^* \cup D_2^*$ and $S \cap V_2 = B_1$ is a separating incompressible annulus in V_2.

Then by Lemma 4.2, B_1 is a ∂-parallel annulus in V_2 and the parallelism contains γ_2. Let X_1 and X_2 be the closure of the two components of $V_1 - (D_1^* \cup D_2^*)$ with $\partial\gamma_1 \subset X_1$, Y_1 and Y_2 the closure of the two components of $V_2 - B_1$ with $\gamma_2 \subset Y_1$ as in Figure 9.

Then $(X_1, X_1 \cap \gamma_1) \cup (Y_1, Y_1 \cap \gamma_2)$ extends to a 2-bridge decomposition of one of K_1 and K_2, say K_1, and $(X_2, X_2 \cap \gamma_1) \cup Y_2$ extends to a genus two Heegaard splitting of some 3-sphere containing K_2 as a core of a handle. Then $t(K_2) = 1$ and we get the conclusion (1) or (2) in our theorem.

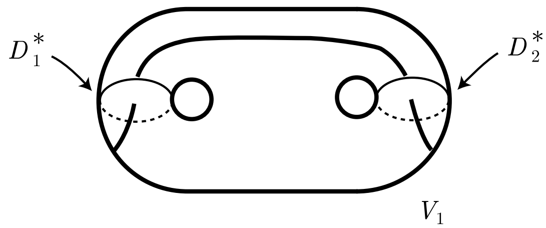

Figure 10

Case II-(3) : $\{D_1^*, D_2^*\}$ is a complete meridian disk system of V_1 as in Figure 10.

Suppose $A_1 \cup A_2 \cup \cdots \cup A_\ell \neq \emptyset$. Then, since $D_1^* \cup D_2^*$ cuts open V_1 into a 3-ball, $A_1 \cup A_2 \cup \cdots \cup A_\ell$ are all compressing annuli. Then we have a contradiction as in the proof of Case I-(2). Hence $S \cap V_1 = D_1^* \cup D_2^*$. This means that S is a non-separating 2-sphere in S^3, and this contradiction shows that Case II-(3) does not occur and completes the proof of our Theorem 1.6 □.

Proof of Proposition 1.7 Let (V_1, V_2) be a Heegaard splitting of a 3-sphere S_1^3 which realizes the tunnel number of K, i.e., V_1 contains K as a core of a handle of V_1 and $g(V_1) = t(K) + 1 = g_1(K)$. Let $(B_1, \gamma_1 \cup \delta_1)$ and $(B_2, \gamma_2 \cup \delta_2)$ be a 2-bridge decomposition of K' in another 3-sphere S_2^3, i.e., $(B_i, \gamma_i \cup \delta_i)$ is a 2-string trivial tangle $(i = 1, 2)$ and $K' = \gamma_1 \cup \gamma_2 \cup \delta_1 \cup \delta_2 \subset B_1 \cup B_2 = S_2^3$.

Let D be a meridian disk of V_1 which intersects K in a single point and $N(D)$ a regular neighborhood of D in V_1, then we can put $N(D) = D \times [0, 1]$ and $N(D) \cap K = x \times [0, 1]$, where x is a point in Int$D$. Let $N(\delta_2)$ be a regular neighborhood of δ_2 in B_2, then we can put $N(\delta_2) = D' \times [0, 1]$ and $\delta_2 = y \times [0, 1]$, where D' is a disk and y a point in IntD'.

Let $K \# K'$ be the connectd sum of K and K'. Then $K \# K'$ is a knot in the 3-sphere $S^3 = cl(S_1^3 - N(D)) \cup_{\partial N(D) = \partial N(\delta_2)} cl(S_2^3 - N(\delta_2))$. Put $W_1 = cl(V_1 - N(D))$. Then, since $N(D) \cap W_1 = \partial N(D) \cap \partial W_1 = D \times \{0, 1\}$ and since $N(\delta_2) \cap B_1 = \partial N(\delta_2) \cap \partial B_1 = D' \times \{0, 1\}$, W_1 and B_1 is glued along the two disks $D \times \{0, 1\} = D' \times \{0, 1\}$. Hence $U_1 = W_1 \cup_{D \times \{0,1\} = D' \times \{0,1\}} B_1$ is a genus $g_1(K)$ handlebody and $(K \# K') \cap U_1$ is a trivial arc in U_1 because $(K \# K') \cap W_1$ is a trivial arc in W_1 and $(K \# K') \cap B_1 \subset B_1$ is a 2-string trivial arc in B_1.

On the other hand, put $W_2 = cl(B_2 - N(\delta_2))$. Then, since $N(D) \cap V_2 = \partial N(D) \cap \partial V_2 = \partial D \times [0, 1]$ and since $N(\delta_2) \cap W_2 = \partial N(\delta_2) \cap \partial W_2 = \partial D' \times [0, 1]$, V_2 and W_2 is glued along the annulus $\partial D \times [0, 1] = \partial D' \times [0, 1]$. Hence $U_2 = V_2 \cup_{\partial D \times [0,1] = \partial D' \times [0,1]} W_2$ is a genus

$g_1(K)$ handlebody and $(K\#K')\cap U_2$ is a trivial arc in U_2 because δ_2 is a trivial arc in B_2 and $(K\#K')\cap W_2$ is a trivial arc in W_2.

Hence (U_1, U_2) is a genus $g_1(K)$ Heegaard splitting of S^3 which gives a 1-bridge decomposition of $K\#K'$. This implies $g_1(K\#K') \leq g_1(K)$ and completes the proof of Proposition 1.7 □.

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